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(6) GLOBAL SOLUTIONS OF SEMILINEAR EVOLUTION EQUATIONS SATISFYING AN ENERGY INEQUALITY,

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GLOBAL SOLUTIONS OF SEMILINEAR EVOLUTION EQUATIONS SATISFYING AN ENERGY INEQUALITY

Hansjörg Kielhöfer

Technical Summary Report #1869
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ABSTRACT

We prove the global existence (in time) for any solution of an abstract semilinear evolution equation in Hilbert space provided the solution satisfies an energy inequality and the nonlinearity does not exceed a certain growth rate. When applied to semilinear parabolic initial-boundary-value problems the result admits also the limiting growth rates which were given by Sobolevskii and Friedman, but which were not permitted in their theorem. The Navier-Stokes system in two dimensions is a special case of our general result. The method is based on the theories of semigroups and fractional powers of regularly accretive linear operators and on a nonlinear integral inequality which gives the crucial a-priori estimate for global existence.

AMS(MOS) Subject Classifications: 34G05, 35K45, 35K50, 35K55, 47F05, 47H15.

Key Words: Semilinear parabolic initial-boundary value problem, Global solution, Energy inequality, Evolution equation, Initial value problem in Hilbert space, Semigroups and fractional powers of regularly accretive operators.

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SIGNIFICANCE AND EXPLANATION

Semilinear parabolic initial-boundary-value problems occur very often in applications. Typical examples are the basic equations in hydrodynamics (Navier-Stokes system), heat conduction, and reaction-diffusion processes.

Given any initial value (i.e. an initial velocity field, an initial temperature distribution and so on) the parabolic partial differential equations describe the evolution of this initial status in time under certain time independent boundary conditions. If these equations are linear they do not allow a "blowing up" of the solutions in finite time, thus describing the process for all time in the future.

If there are nonlinear terms, however, a typical phenomenon is the possibly non-global existence of a solution - the solution may exist only for a certain finite length of time. This can be seen already with simple examples in ordinary differential equations.

In addition to the partial differential equation, however, there often exist upper bounds for the energy of the system, which is expressed by an energy inequality. Such an energy inequality will suffice to assure the global existence for solutions of ordinary differential equations. The situation is different for partial differential equations of parabolic type: a partial derivative (with respect to the space variable for instance) might blow up without increasing the energy to infinity. If this happens the solution ceases to exist. This phenomenon in hydrodynamics, for example, is called "turbulence" and it is not yet completely understood.

If, however, in addition to an energy inequality certain growth conditions on the nonlinearity are fulfilled, such a blowing up as described above can be excluded. A well known example is the Navier-Stokes system in two space dimensions. In this paper we give a general theorem which improves the known results since it admits also the limiting growth rates which were not allowed to be assumed under similar conditions so far. The Navier-Stokes system in two dimensions is a special case of our general result which allows many other applications.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

GLOBAL SOLUTIONS OF SEMILINEAR EVOLUTION EQUATIONS

SATISFYING AN ENERGY INEQUALITY

Hansjörg Kielhöfer

Introduction

A well known functional analytic approach to semilinear parabolic initial-boundary value problems of the type

$$(0.1)\qquad \begin{aligned} &u_t + \sum_{|\tilde\alpha|\le 2m} a^{\tilde\alpha}(t,x)D_{\tilde\alpha}u = f(t,x,D_{\tilde\gamma_1}u,\ldots,D_{\tilde\gamma_\ell}u) \text{ in } (0,T)\times\Omega,\ 0<T\le\infty,\ \Omega\subset\mathbb{R}^n,\\ &B_j u = 0 \text{ on } (0,T)\times\partial\Omega,\ j=0,\ldots,m-1,\\ &u(0,x) = u_0(x) \text{ in } \Omega, \end{aligned}$$

is briefly described as follows:

The unknown solution $u(t,\cdot) = u(t)$ is considered as a curve in some appropriate space E of functions on Ω. As such it is to satisfy a corresponding initial value problem in E:

$$(0.2)\qquad \begin{aligned} &\frac{du}{dt} + A(t)u = F(t,u), \quad t\in(0,T),\\ &u(0) = u_0. \end{aligned}$$

By A we denote the family of linear differential operators which for $t\in[0,T)$ is given by

$$(0.3)\qquad A(t) \equiv \sum_{|\tilde\alpha|\le 2m} a^{\tilde\alpha}(t,x)D_{\tilde\alpha}.$$

The boundary conditions are replaced by the condition

$$(0.4)\qquad u(t)\in D(A),\quad t\in(0,T),$$

where $D(A)$ denotes the time-independent domain of definition of the family A.

Parabolicity means that this family A is uniformly elliptic on every closed interval $[0,T_0]\subset[0,T)$, and the semilinearity is expressed by the fact that

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$$F(t,u) = f(t,x,D_{\tilde{\gamma}_1} u, \ldots, D_{\tilde{\gamma}_\ell} u) \tag{0.5}$$

is of lower order then $A(t)$, i.e. $|\tilde{\gamma}_\kappa| \leq 2m - 1$.

In Section 2 we shall render our assumptions on A, F, and the boundary conditions B_j much more precise.

Of course, the notion of a solution is different for problems (0.1) and (0.2), and, in general, the solutions are not equivalent.

A solution of (0.1) is called a classical solution if all derivatives occuring in the partial differential equation exist in the classical sense and are continuous in $(0,T) \times \Omega$, and the equation, the boundary conditions, and the initial condition are fulfilled pointwise.

A solution of (0.2) is called a strict solution if du/dt exists in the strong topology of E, if $u(t)$ is in the domain $D(A)$ for all $t \in (0,T)$, if all terms in the differential equation are continuous on $(0,T)$ (in the strong topology of E), and if u is also continuous at $t = 0$ and fulfills the initial condition. (This definition tacitly implies that the nonlinearity F makes sense for all u in the domain $D(A)$).

From the point of view of existence of solutions it is desirable to be able to show that a solution of (0.2) is in fact a classical solution of (0.1). This turns out to be the case if the data of problem (0.1) are smooth enough (see Section 2).

The local existence theory for problem (0.2) is quite analogous to that of ordinary differential equations in finite dimensional spaces. Although unbounded operators are involved in equation (0.2), they disappear if it is rewritten as a Volterra integral equation in some appropriate subspace of E (which is the domain of some fractional power of A). That integral equation, however, has a weak singularity.

The main tools of this approach are the theories of semigroups and fractional powers of linear operators. Among many others who made considerable contributions to these theories we want to mention T. Kato and P. E. Sobolevskii who were in turn influenced by K. Yosida and M. A. Krasnosel'skii respectively.

Our approach is closely related to Sobolevskii's article [23] (see also Friedman's book [7]). For the applications he chose $E = L_p(\Omega)$ where Ω has to be bounded. If Ω

is unbounded we refer to [12], [13]. In addition to $E = L_p(\Omega)$ we also chose the Hölder space $E = C^{\mu}_{*}(\bar{\Omega})$ where the star indicates some decay for $|x| \to \infty$ which is weaker than that in $L_p(\Omega)$ (see Section 2). Thus solutions of (0.2) can be found which are not necessarily in $L_p(\Omega)$. (In addition to that, the C^{μ}-approach immediately gives, by definition, classical solutions.)

In complete analogy to ordinary differential equations the local solutions of (0.2) don't necessarily exist on the whole time interval $(0,T)$. By a continuation argument, global existence follows from an a-priori estimate of the solution $u(t)$. But in contrast to the finite dimensional case an a-priori bound in E won't suffice in general to guarantee global existence. (For a counter-example see [15], p. 39. Unfortunately this example doesn't fit into the framework of this paper since the boundary conditions depend on the time t.) Thus, in addition to an a-priori bound in E, certain growth conditions on the non-linearity are required.

The first general results in this connection are due to Sobolevskii [24] and Friedman [6]:

Theorem 0.1. Assume that for some $1 < q \leq \infty$ every solution of (0.2) satisfies $\|u(t)\|_{L_q(\Omega)} \leq C(T_0)$ on $[0,T_0] \subset [0,T)$ and the nonlinearity satisfies

$$(0.6) \qquad |f(t,x,D_{\tilde{\gamma}_1} u,\dots,D_{\tilde{\gamma}_\ell} u)| \leq C(1 + \sum_{\kappa=1}^{\ell} |D_{\tilde{\gamma}_\kappa} u|^{r_\kappa}) \quad , \text{ where}$$

$$(0.7) \qquad r_\kappa < \frac{2m + (n/q)}{|\tilde{\gamma}_\kappa| + (n/q)} = R(|\tilde{\gamma}_\kappa|) \ .$$

Then any solution of (0.2) in $L_p(\Omega)$ (for some $p > n$) exists on $(0,T_0]$. (Observe that any solution of (0.2) in $L_p(\Omega)$ for some $p > n$ actually is in $L_q(\Omega)$ for all $q \leq \infty$.)

The method of their proof can't be generalized to allow a growth rate $r_\kappa = R(|\tilde{\gamma}_\kappa|)$. (See the proof of Theorem 2.2.) The key point is to obtain an a-priori bound in a norm which is strong enough to estimate the nonlinearity. Thus the semilinear problem can (formally) be considered to be a linear equation with a bounded inhomogeneity.

The cases of Theorem 0.1 most often used in applications are $q = 2$ and $q = \infty$. Whereas the latter applies essentially only to second order equations (maximum principle), an L_2-a-priori bound can be obtained in a reasonably large class of problems of arbitrary order $2m$. A sufficient condition is given in, e.g., [24], [6]:

$$\text{(0.8)} \qquad \operatorname{Re} f(t,x,D_{\tilde{\gamma}_1}u,\dots,D_{\tilde{\gamma}_\ell}u)\bar{u} \le a|u|^2 + b \quad , \quad a,b \ge 0 \quad ,$$

provided $\operatorname{Re}(A(t)u,u) \ge 0$. (Here and in the following $(\ ,\ )$ and $\|\ \|$ denote the scalar product and norm in $L_2(\Omega)$.)

It is much more general to give (0.8) in the form which it is needed in:

$$\text{(0.9)} \qquad \operatorname{Re}(f(t,x,D_{\tilde{\gamma}_1}u,\dots,D_{\tilde{\gamma}_\ell}u)\ ,u) \le a\|u\|^2 + b$$

for all $u \in D(A)$ and $(t,x) \in [0,T_0] \times \Omega$.

One of the best known examples where (0.9) is fulfilled is given by the Navier-Stokes system written as an evolution equation (0.2) in a suitable Hilbert space $E \subset L_2(\Omega)$ (see [22], [11], [8]).

It is also well known that any strict solution exists globally provided it is a plane flow $(n = 2)$. But since the nonlinearity in this case has the form uD_1u (D_1 is a first order differential operator) and $R(0) = 3$, $R(1) = 3/2$ for $n = 2, q = 2$, this global existence result is <u>not</u> a consequence of Theorem 0.1.

Where does the additional information come from? It comes from an energy inequality

$$\text{(0.10)} \qquad \|u(t)\|^2 + \operatorname{Re}\int_0^t (A(s)u(s),u(s))ds \le C(T_0) \quad \text{on} \quad [0,T_0] \quad .$$

(The constant $C(T_0)$ depends also on other data of the problem like the initial condition, external forces, the domain Ω and so on. But we consider these quantities to be fixed.)

Such an energy inequality (0.10) is valid for the Navier-Stokes system. We emphasize that in most cases where an L_2-a-priori estimate is obtained also an energy inequality is valid. (It certainly is under the assumptions in [24], [6].) Therefore it is more natural to assume (0.10) instead of only $\|u(t)\| \le C(T_0)$.

The first general global existence theorem for plane flows, after the pioneering work of J. Leray in the years 1933-34, was given by O. A. Ladyshenskaja in 1958 [14]. We formally sketch her proof from the point of view of possible generalizations. Let E be the closure in $L_2(\Omega)$ of all smooth solenoidal $(\operatorname{div} u = 0)$ vector-valued functions with compact support in Ω and let P be the orthogonal projector on E. Then $A = -P\Delta$ can be given a dense domain, generalizing homogeneous Dirichlet boundary conditions, such that it is a positive self-adjoint operator in E. The nonlinearity is given by $F(t,u) = -P[(u\cdot\nabla)u] + g(t)$, where g comes from the external force. With these definitions we are led to the abstract initial value problem (0.2) in E. In view of the self-adjointness we have $(Au,u) = \|A^{1/2}u\|^2$.

After differentiating the equation with respect to t and scalar multiplication by u_t Ladyshenskaja obtains

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \|A^{1/2}u_t\|^2 = -((u_t\cdot\nabla)u,u_t) + (g_t,u_t) \quad , \tag{0.11}$$

where $\operatorname{div} u_t = 0$ is used. Since $\|u_t\|^2_{L_4(\Omega)} \leq C\|u_t\|\ \|\nabla u_t\|$ is valid in two dimensions, she has the estimate

$$((u_t\cdot\nabla)u,u_t) \leq C\|u_t\|\ \|\nabla u\|\ \|\nabla u_t\| \quad ,$$

and so

$$\frac{d}{dt}\|u_t\|^2 + 2\|A^{1/2}u_t\|^2 \leq C^2\|A^{1/2}u\|^2\|u_t\|^2 + \|A^{1/2}u_t\|^2 + \|u_t\|^2 + \|g_t\|^2 \quad .$$

This implies, in view of (0.10), a bound on $\|u_t(t)\|$ on $[0,T_0]$. (We used the equality $\|\nabla u\| = \|A^{1/2}u\|$.) The original differentiated version of the energy inequality (0.10), namely

$$(u_t,u) + \|A^{1/2}u\|^2 \leq a\|u\|^2 + b \quad \text{on} \quad [0,T_0] \quad , \tag{0.12}$$

finally implies an a-priori estimate of $\|A^{1/2}u(t)\|$ which is strong enough to guarantee the global existence of the strict solution of (0.2).

Looking toward generalizations we first observe that the self-adjointness of A is not necessary for her proof. Furthermore it applies to any nonlinearity which satisfies an estimate like

$$(0.13)\quad \mathrm{Re}\left(\frac{d}{dt}F(t,u(t)),\ u_t(t)\right) \le \tilde{C}(T_0)\left(1 + \|u_t(t)\|^2 + \|A^{1/2}u(t)\|^2\|u_t(t)\|^2\right) + \|A^{1/2}u_t\|^2$$

for all $u(t)$ such that $\|u(t)\| \le C(T_0)$ on $[0,T_0]$. (Observe that in the case of the Navier-Stokes system an estimate like $\|(u_t\cdot\nabla)u\| \le C\|\nabla u\|\ \|\nabla u_t\|$ is not true in two dimensions.) The assumption (0.13) refers to the total derivative of F with respect to t and thus to the partial derivatives of f, if F is given by (0.5). A closer analysis shows that it allows growth rates equal to $R(|\tilde{\gamma}_\kappa|) - 1$ for the derivatives of f with respect to $D_{\tilde{\gamma}_\kappa} u$, provided $|\tilde{\gamma}_\kappa| \le m$. Adding sign-conditions on the derivatives of f, even bigger growth rates can be admitted. In this connection we refer to the results of [19], [20], whose approach is in this spirit, but different in so far as they impose a sign-condition on a primitive of f which must not depend on a derivative of u or on (t,x) explicitely.

In case of the Navier-Stokes system the estimate (0.13) on the derivative of F does not influence the quadratic nonlinearity $-P[(u\cdot\nabla)u]$. But, however, it requires additional regularity of the external force g, namely $g_t \in L_2((0,T),E)$.

In the year 1959 Sobolevskii [22] gave a global existence proof without requiring any differentiability of g. Since our results are closely related to Sobolevskii's approach we also briefly sketch his proof. First of all he established the estimates

$$(0.14)\qquad \|A^{-\eta}P[(u\cdot\nabla)u]\| \le C(\eta)\|A^{(1/2)-\eta}u\|\ \|A^{1/2}u\|,\quad 0 < \eta < \frac{1}{2}\ .$$

Using Heinz' inequality for fractional powers of self-adjoint operators he proved (0.14) replacing A by $-\Delta$ and cancelling the projector P. In this case he expressed $(-\Delta)^{-\eta}$ in terms of the Green's function of the first boundary value problem for the Laplacian, which finally led to the desired estimate. Note that (0.14) is not valid for $\eta = 0$. Continuing he used the Volterra integral equation related to (0.2):

$$(0.15)\qquad u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}F(s,u(s))ds\ .$$

From (0.15) he derived two estimates:

$$(0.16) \qquad \|A^{1/4}u(t)\| \le \phi(t) + C(1/4)\left(\int_0^t \|A^{1/4}u(s)\|^2\|A^{1/2}u(s)\|^2 ds\right)^{1/2} ,$$

$$(0.17) \qquad \|A^{1/2}u(t)\| \le \psi(t) + C(1/4)\int_0^t (t-s)^{-3/4}\|A^{1/4}u(s)\|\,\|A^{1/2}u(s)\|ds ,$$

where

$$\phi(t) = \|A^{1/4}u_0\| + \|A^{1/4}\int_0^t e^{-A(t-s)}g(s)ds\| ,$$

$$\psi(t) = \|A^{1/2}u_0\| + \|A^{1/2}\int_0^t e^{-A(t-s)}g(s)ds\| .$$

Assuming only $g \in L_2((0,T), E)$, in view of the energy inequality, the relation (0.16) implies an estimate on $\|A^{1/4}u(t)\|$ which, via (0.17), finally gives the desired bound for $\|A^{1/2}u(t)\|$. (For the most general assumptions on u_0 and g allowing (0.15) to be solved globally see [11], [8].)

Since no compactness is used, Sobolevskii's as well as Ladyshenskaja's proofs hold also for unbounded domains. The conditions on g which imply that u is actually a solution of the initial value problem (0.2) or moreover a classical solution can be found in [8].

The key of Sobolevskii's proof - besides the remarkable estimates (0.14) - is the inequality

$$(0.18) \qquad \|A^{1/2}\int_0^t e^{-A(t-s)}g(s)ds\| \le \left(\int_0^t \|g(s)\|^2 ds\right)^{1/2} \quad \text{for all } g \in L_2((0,t), E) .$$

Since A is a positive self-adjoint operator (0.18) is simply proved by considering $\frac{du}{dt} + Au = g$, $u(0) = 0$, and multiplying the equation by Au.

A generalization of the global existence result sketched above to non-self-adjoint operators A requires an inequality analogous to (0.18). In [27] Sobolevskii states for operators A generating an exponentially decreasing analytic semigroup the following inequality

(0.19) $$\left\|\int_0^t e^{-A(t-s)}g(s)ds\right\|_{1/2} \leq c\left(\int_0^t \|g(s)\|^2 ds\right)^{1/2} ,$$

where $\|u\|_{1/2} = (\int_0^\infty \|Ae^{-As}u\|^2 ds)^{1/2}$. At that time only the estimate $\|A^\gamma u\| \leq c\|u\|_{1/2}$ for $0 \leq \gamma < \frac{1}{2}$ was known to him. To prove the equivalence of the norms $\|A^{1/2}\cdot\|$ and $\| \ \|_{1/2}$ the following techniques are used: First of all $D(A^{1/2})$ is an "interpolation space" (or "intermediate space" or "espace de trace") between $D(A)$ and E (with equivalent norms), provided A is "regularly accretive" (see [16]). The different notions of intermediate spaces are equivalent and the norm $\| \ \|_{1/2}$ is equivalent to one of these norms (see [5], chap. III, Theorem 3.5.3). Unfortunately Sobolevskii gave no proof of (0.19), so we shall not use it.

The goal of this paper is to generalize Sobolevskii's global existence result to a class of abstract semilinear evolution equations in Hilbert space admitting an energy inequality. We allow A to depend on the time t and $A(t)$ is regularly accretive (and not necessarily self-adjoint) for fixed t. (These operators are the appropriate generalizations of self-adjoint operators since they arise in "variational problems"; see Section 2.) Since we don't use an inequality like (0.18) or (0.19) we give a different proof to estimate $\|A^{1/2}u(t)\|$.

Whereas all results described so far are obtained using linear differential inequalities (Ladyshenskaja) or integral inequalities (Sobolevskii, Friedman), we derive the crucial estimate for $\|A^{1/2}u(t)\|$ by a nonlinear integral inequality (see Lemma 1.2).

When applied to semilinear parabolic problems our results improve Theorem 0.1 allowing for f also the limiting growth rates $r_\kappa = R(|\tilde{\gamma}_\kappa|)$, provided $q = 2$ and $|\tilde{\gamma}_\kappa| \leq m$. Thus the Navier-Stokes system in two dimensions is no longer a special or "singular" case but embedded into a general Theorem.

Our results will also improve Theorem II.1 in [28] with respect to the assumptions on the nonlinearity and initial conditions. Since our notion of a solution is stronger than that in [28] we require, however, Hölder continuity of the inhomogeneity but we admit a non-square integrable singularity at $t = 0$. (Theorem I.1 in [28] is a consequence of

Sobolevskii's result: indeed, the same simple argument which proves (0.18) in the case of a positive self-adjoint operator also shows the inequality

$$\int_0^T \left\| A\int_0^t e^{-A(t-s)} g(s)\,ds \right\|^2 dt \leq \int_0^T \|g(s)\|^2 ds \quad . \tag{0.20}$$

Thus, for any $u_0 \in D(A^{1/2})$ the solution of (0.15) in the sense of [22] or [8], [11] is actually a strong solution in the sense of [28] satisfying an a-priori estimate as indicated in [28], Remark I.1.

In Section 1 we prove the abstract result in Hilbert space. Applications and possible generalizations will be given in Section 2.

We want to express our gratitude to A. Pazy for many helpful discussions which led to the final assumptions on $A(t)$.

Section 1

The goal of this section is to formulate and to prove Theorem 1.1 which states the main result for the initial value problem (0.2) in a Hilbert space H (by $c_1, c_2, \ldots$ we denote positive constants depending on quantities as indicated.)

Let $V \subset H$ be two Hilbert spaces with scalar products $(\ ,\ )_m$, $(\ ,\ )$ and norms $\|\ \|_m$, $\|\ \|$ respectively. (The subscript m arises from the applications in Section 2.) We assume that V is dense in H and that the embedding is continuous.

Let $A = \{A(t),\ t \in [0,T)\}$, $0 < T \leq \infty$, be a family of closed linear operators satisfying

(A1) The domain $D(A) \subset V \subset H$ of $A(t)$ is time independent and dense in H;

(A2) $\mathrm{Re}(A(t)u,u) \geq c_1(T_0)\|u\|_m^2 - k\|u\|^2$
for all $t \in [0,T_0] \subset [0,T]$, $u \in D(A)$, and with $c_1(T_0) > 0$, $k \geq 0$;

(A3) $|(A(t)u,u)| \leq c_2(T_0)\|u\|_m^2$
for all $t \in [0,T_0]$, $u \in D(A)$, *and some* $c_2(T_0) > 0$;

(A4) $P(A(t)) \cap \{\lambda \in \mathbb{C},\ \mathrm{Re}\,\lambda < 0\} \neq \emptyset$
for all $t \in [0,T)$, where P denotes the resolvent set;

(A5) there exists a Hilbert space $Y \subset H$ such that

(i) V is a closed subspace of $[Y,H]_{1/2}$ (the interpolation space of order $1/2$ between Y and H),

(ii) $D(A) \subset Y$, $D(A^*(t)) \subset Y$ for all $t \in [0,T)$, where $A^*(t)$ is the adjoint of $A(t)$;

(We can replace (A5) by

(A5)' $D(A^*(t)) = D(A)$ for $t \in [0,T)$.)

(A6) $\|A(t_1)u - A(t_2)u\| \leq c_3(T_0)(\|A(s)u\| + \|u\|)|t_1-t_2|^{\alpha}$,
for all $t_1, t_2, s \in [0,T_0]$, $u \in D(A)$, and some $c_3(T_0) > 0$, $0 < \alpha \leq 1$.

We collect some consequences of these assumptions. First of all, in assumption (A2) we can assume without loss of generality that $k = 0$ by simply adding ku on both sides of equation (0.2). Then assumptions (A1) to (A4) imply that for any fixed $t \in [0,T)$ the operator $A(t)$ is regularly accretive in the sense of [9], [16] (see the arguments in Section 2).

Thus $-A(t)$ generates an analytic semigroup $e^{-\tau A(t)}$, $\tau \geq 0$, on H. This result, which is due to Kato [9], is a consequence of the following more general lemma:

Lemma 1.1 Let B be a closed operator with dense domain $D(B)$ in H. Define

$$S(B) = \{\|u\|^{-1}(Bu,u),\ 0 \neq u \in D(B)\} \subset \mathbb{C} \ .$$

Then:

(i) If $\mathrm{dist}(\lambda,S(B)) > 0$ then $B - \lambda I$ is injective and has closed range. (I denotes the identity, dist the distance from λ to $S(B)$.)

(ii) Let P_0 be a component of the complement of the closure $\overline{S(B)}$ in $\mathbb{C}$ with the property $P_0 \cap P(B) \neq \emptyset$. Then $P_0 \subset P(B)$ and moreover

$$\|(B - \lambda I)^{-1}\| \leq \mathrm{dist}(\lambda,S(B))^{-1} \quad \text{for all } \lambda \in P_0 \ .$$

We give a proof of Lemma 1.1 which we owe to A. Pazy in the Appendix.

Now, by (A2), (A3), we have for $t \in [0,T_0]$

$$c_2(T_0)c_1(T_0)^{-1}\, \mathrm{Re}(A(t)u,u) \geq |\mathrm{Im}(A(t)u,u)| \ ,$$

which implies that

$$S(A(t)) \subset S_{\theta_1} = \{\lambda \in \mathbb{C},\ -\theta_1 < \arg\lambda < \theta_1\} \quad \text{for some } 0 < \theta_1(T_0) < \frac{\pi}{2} \ .$$

Choose $\theta_1 < \theta_2 < \frac{\pi}{2}$ and set

$$S_{\theta_2} = \{\lambda \in \mathbb{C},\ -\theta_2 < \arg\lambda < \theta_2\} \ .$$

Let P_{θ_2} be the complement of $\overline{S}_{\theta_2}$. Then there exists a constant $c_4 = c_4(\theta_2)$ such that

$$\mathrm{dist}(\lambda,\ S(A(t))) \geq c_4(T_0)|\lambda| \quad \text{for all } t \in [0,T_0],\ \lambda \in P_{\theta_2} \ .$$

By assumption (A4) we have $P(A(t)) \cap P_{\theta_2} \neq \emptyset$, hence by the previous lemma $-A(t)$ is a generator of an analytic semigroup on H (see e.g. [7]).

By adding $\tilde{k}u$ on both sides of (0.2), where $\tilde{k} > k + d$, we can assume without loss of generality that

(1.1) $$\|e^{-\tau A(t)}\| \leq c_5(T_0)e^{-dt}, \quad \tau \geq 0,\ t \in [0,T_0],\ d > 0 \ .$$

Thus $0 \in P(A(t))$ for all $t \in [0,T)$ and assumption (A6) implies

(1.2) $$\|A(t_1)A^{-1}(t_2)\| \leq c_6(T_0) \ ,$$

(1.3) $$\|(A(t_1) - A(t_2))A^{-1}(s)\| \leq c_7(T_0)\,|t_1-t_2|^{\alpha},\quad t_1,t_2,s \in [0,T_0] \ .$$

Properties (1.2) - (1.3) suffice to show the existence of a "fundamental solution" $U(t,s)$, $0 \leq s \leq t \leq T_0$ of the linear homogeneous equation $\frac{du}{dt} + A(t)u = 0$. $U(t,s)$ is a family of bounded linear operators defined on H which is continuous with respect to t and s in the strong topology of $L(H,H)$, the space of bounded linear operators from H into H. Furthermore $U(t,s)$ has the following properties:

(1.4) $$\frac{\partial}{\partial t}U(t,s) + A(t)U(t,s) = 0 \ , \quad 0 \leq s < t \leq T_0 \ ,$$

(1.5) $$\frac{\partial}{\partial t}U(t,s) \text{ is continuous in } L(H,H) \text{ for } 0 \leq s < t \leq T_0 \ ,$$

(1.6) $$U(t,t) = I$$

(1.7) $$U(t,s)U(s,\tau) = U(t,\tau) \ , \quad 0 \leq \tau \leq s \leq t \leq T_0 \ ,$$

(1.8) $$\lim_{h \downarrow 0} \left\|\frac{1}{h}(U(t+h,\, t)u - u) - (-A(t)u)\right\| = 0 \ , \quad u \in D(A) \ .$$

For a proof of the existence of $U(t,s)$ and of the properties listed above we refer to [23] or [7].

In view of (1.1), fractional powers of $A(t)$ can be defined for each $t \in [0,T)$ as follows:

$$(1.9)\qquad A^{-\gamma}(t) = \Gamma(\gamma)^{-1}\int_0^\infty e^{-\tau A(t)}\tau^{\gamma-1}d\tau,\ \gamma > 0 \quad ,$$

$$(1.10)\qquad A^{\gamma}(t) = (A^{-\gamma}(t))^{-1},\ D(A^{\gamma}(t)) = R(A^{-\gamma}(t)) \quad ,$$

where $R(A^{-\gamma}(t))$ denotes the range of the everywhere defined operator $A^{-\gamma}(t)$. For $0 \le \beta \le \gamma \le 1$ these fractional powers satisfy the inequality of moments (or interpolation inequality):

$$(1.11)\qquad \|A^{\beta}(t)u\| \le c_8(T_0,\beta,\gamma)\|A^{\gamma}(t)u\|^{\beta/\gamma}\|u\|^{1-(\beta/\gamma)},\ u \in D(A^{\gamma}(t)),\ t \in [0,T_0] \quad ,$$

where $A^0(t) = I$ by definition.

Furthermore, as regards the semigroup and the fundamental solution, the following estimates are valid:

$$(1.12)\qquad \|A^{\gamma}(t)e^{-\tau A(t)}\| \le c_9(T_0,\gamma)e^{-d\tau}\tau^{-\gamma},\ \tau > 0,\ t \in [0,T_0] \quad ,$$

$$(1.13)\qquad \|A^{\gamma}(\theta)U(t,s)\| \le c_{10}(T_0,\gamma)(t-s)^{-\gamma},\ 0 \le s < t \le T_0,\ \theta \in [0,T_0] \quad ,$$

$$(1.14)\qquad \|A^{\gamma}(t)U(t,s)A^{-\beta}(s)\| \le c_{11}(T_0,\gamma,\beta)(t-s)^{\beta-\gamma},\ 0 \le s < t \le T_0,\ 0 \le \beta \le \gamma < 1 + \alpha \quad ,$$

where α is the Hölder exponent in (A6) or (1.3). Analyzing the method of construction of $U(t,s)$ we get the following estimate

$$(1.15)\qquad \|A^{\gamma}(s)U(t,s)u\| \le c_{12}(T_0,\gamma,\eta)(t-s)^{-\gamma-\eta}\|A^{-\eta}(s)u\|,\ u \in H ,$$

$$0 \le s < t \le T_0,\ 0 \le \gamma < 1,\ 0 \le \eta < \alpha \quad .$$

Finally we shall need

$$(1.16)\qquad \|A^{\gamma}(\theta)(U(t+h,\ s) - U(t,s))A^{-\beta}(s)\| \le c_{13}(T_0,\gamma,\beta)h^{\delta-\gamma}(t-s)^{\beta-\delta} \quad ,$$

$$0 \le s < t < t+h \le T_0,\ 0 \le \gamma \le 1,\ 0 \le \beta \le \delta < 1 + \alpha,\ 0 < \delta - \gamma \le 1 \quad .$$

For all details we refer again to [23] or [7].

Definitions (1.9), (1.10) do not exclude a time dependence of the domains $D(A^{\gamma}(t))$ for $0 < \gamma < 1$. But a remarkable result of Kato [10], generalizing the Heinz inequality,

applies to our case: Properties (A1), (1.2) together with the regular accretiveness of $A(t)$ (= maximal accretiveness in [10]) imply that the domains $D(A^{\gamma}(t))$ are actually independent of t and that

$$(1.17) \qquad \|A^{\gamma}(t_1)A^{-\gamma}(t_2)\| \le c_{14}(T_0,\gamma) \quad , \quad t_1,t_2 \in [0,T_0] \quad ,$$

holds for all $\gamma \in [0,1]$. Therefore we shall write $D(A^{\gamma})$ for $D(A^{\gamma}(t))$. Furthermore, under the assumptions (A5) (or (A5)') the domain $D(A^{1/2})$ can be given explicitly:

$$(1.18) \qquad D(A^{1/2}) = V \quad .$$

Moreover

$$(1.19) \qquad c_{15}(T_0)\|A^{1/2}(t)u\| \le \|u\|_m \le c_{16}(T_0)\|A^{1/2}(t)u\|, \; t \in [0,T_0] \quad ,$$

as shown by Lions [16], who strongly used Kato's result mentioned above. (It is not too difficult to prove a continuous embedding $D(A^{\gamma}) \subset V$ for $\gamma > \frac{1}{2}$. The delicate problem is to prove that $\gamma = \frac{1}{2}$ is admitted also. There the regular accretiveness is needed.) Lions showed that $D(A^{\gamma})$ is an interpolation space between $D(A)$ and H. We shall use this later in Section 2.

In view of (1.17) we can derive from (1.14)

$$(1.20) \qquad \|A^{\gamma}(\theta)U(t,s)A^{-\beta}(\theta)\| \le c_{17}(T_0,\gamma,\beta)(t-s)^{\beta-\gamma}, \; \theta \in [0,T_0] \quad ,$$

and finally (1.15) implies the estimate

$$(1.21) \qquad \|A^{\gamma}(\theta)U(t,s)u\| \le c_{18}(T_0,\gamma,\eta)(t-s)^{-\gamma-\eta}\|A^{-\eta}(\theta)u\| \quad ,$$

for $u \in H, \; \theta \in [0,T_0], \; 0 \le \gamma < 1, \; 0 \le \eta < \alpha$.

We are now ready to formulate our assumptions on the nonlinearity F. At first sight they might look artificial, but they are motivated by the applications (see Theorem 2.1).

Let $F:(0,T) \times D(A^{\gamma}) \to H, \; \frac{1}{2} \le \gamma < 1,$ be an operator which we decompose as $F(t,u) = F_0(t,u) + g(t)$. We assume that $g \in C^{\alpha_1}((0,T),H)$ for some $0 < \alpha_1 \le 1$, which

means that g is uniformly Hölder-continuous on any closed interval $[t_0,T_0] \subset (0,T)$. The behaviour of g for $t \downarrow 0$ is described in the following assumptions. By $[t_0,T_0]$ we always denote an arbitrary closed interval contained in $(0,T)$, $\psi_1,\psi_2,\ldots$ are positive functions in $C([0,\infty), \mathbb{R}_+)$, $\mu_1,\mu_2,\ldots$ are positive functions in $C([0,T), \mathbb{R}_+)$, θ is any fixed time in $[0,T)$, and M is any positive number.

(F0) $\mathrm{Re}(F_0(t,u), u) \le q\, c_1(T_0)\, \|u\|_m^2 + c_{19}(t_0,T_0)\,\|u\|^2$

for $t \in [t_0,T_0]$, $u \in D(A)$, where $c_1(T_0)$ is the constant in (A2) and $q < 1$;

(F1) $\|F_0(t,u)\| \le c_{20}(t_0,T_0)\,\psi_1(\|A^{\gamma}(\theta)u\|)$

for $t \in [t_0,T_0]$, $u \in D(A^{\gamma})$;

(F2) $\|A^{-\eta_0}(\theta)F_0(t,u)\| \le \mu_1(t)\,\|u\|_m^{2(1-\eta_0)}\,\psi_2(\|u\|)$

for $t \in (0,T)$, $u \in D(A^{\gamma})$, and some $0 \le \eta_0 < \min(\alpha,\frac{1}{2})$; $\|g(t)\| = O(t^{-1+\varepsilon})$ for $t \downarrow 0$, where $\varepsilon > 0$ is arbitrary;

(F3) $\|A^{-\eta}(\theta)F_0(t,u)\| \le c_{21}(t_0,T_0)\,\psi_3(\|A^{\beta}(\theta)u\|)$

for $t \in [t_0,T_0]$ and for all $\frac{1}{2} \le \beta \le \gamma$ where $\eta = \eta(\beta)$ satisfies a relation $\eta = 1 - \rho\beta$ for some $\rho > 1$;

(F4) $\|F_0(t,u)\| \le \mu_2(t)\,\|A^{\gamma_1}(\theta)u\|\,\psi_4(\|u\|_m)$

for $t \in (0,T)$, $u \in D(A^{\gamma_1})$, $\gamma \le \gamma_1 < 1$,

(F5) $\|A^{-\eta_0}(\theta)(F_0(t,u_1) - F(t,u_2))\| \le \mu_3(t)\,\|u_1 - u_2\|_m(\|u_1\|^{1-2\eta_0}\psi_2(\|u_1\|) + \|u_2\|^{1-2\eta_0}\psi_2(\|u_2\|))$

for $t \in (0,T)$, $u_i \in D(A^{\gamma})$;

(F6) $\|F_0(t_1,u_1) - F_0(t_2,u_2)\| \le$

$$\le c_{22}(t_0,T_0,M)\{|t_1 - t_2|^{\alpha_1} + \|A^{\gamma}(\theta)(u_1 - u_2)\|\,\psi_5(\|A^{\gamma}(\theta)u_1\| + \|A^{\gamma}(\theta)u_2\|)\}$$

for $t \in [t_0,T_0]$, $u_i \in D(A^{\gamma})$, $\|A^{\gamma}(\theta)u_i\| \le M$, $0 < \alpha_1 \le 1$;

(F7) $\|F_0(t_1,u_1) - F_0(t_2,u_2)\| \le c_{23}(T_0,M)\{|t_1 - t_2|^{\alpha_1} + \|A^{\gamma_2}(\theta)(u_1 - u_2)\|^{\alpha_2}\}$

for $t_i \in [0,T_0]$, $u_i \in D(A^{\gamma_2})$, $\|A^{\gamma_2}(\theta)u_i\| \le M$, $\gamma_1 \le \gamma_2 < 1$,

$0 < \alpha_1, \alpha_2 \le 1$;

$g \in C^{\alpha_1}([0,T_0], H)$;

(F8) $F_0(t,u) = G(t,u,u)$ where $G : [0,T) \times D(A^{\gamma_2}) \times D(A^{\gamma_2}) \to H$

satisfies for any $M > 0$ the following two conditions:

(i) $\|G(t,u_1,v) - G(t,u_2,v)\| \le c_{24}(T_0,M)\|A^{\gamma_2}(\theta)(u_1 - u_2)\|$

for $t \in [0,T_0]$, $\|A^{\gamma_2}(\theta)u_i\|, \|A^{\gamma_2}(\theta)v\| \le M$;

(ii) $G(t,u,\cdot) : D(A^{\gamma_2}) \to H$ is for any fixed $t \in [0,T)$ and $u \in D(A^{\gamma_2})$ completely continuous.

Remark 1: If (F2) and (F5) hold then $A^{-\eta_0}(\theta)F(t,\cdot)$ can be extended to the whole space $V = D(A^{1/2})$ so that (F2) and (F5) hold for all $u \in V$.

Remark 2: Obviously (F8) is only needed if $\alpha_2 < 1$ in (F7). The reason for (F8) is simply the following: Since we do not assume Lipschitz-continuity, some compactness has to replace it. But that compactness is only needed with respect to those variables in which F_0 is not Lipschitz-continuous. Let for example $F_0 = F_1 + F_2$ where F_1 is (locally) Lipschitz continuous but F_2 is not. Then only $F_2(t,\cdot) : D(A^{\gamma_2}) \to H$ must be completely continuous.

We shall divide all assumptions on F into two classes:

(H1) = {(F0) to (F6)}

(H2) = {(F0), (F2), (F4), (F7), (F8)}.

Then we have

Theorem 1.1. Let the family A satisfy assumptions (A1) to (A6). The the initial value problem

$$\frac{du}{dt} + A(t)u = F(t,u) \tag{1.22}$$

$$u(0) = u_0$$

possesses a global strict solution on $[0,T)$

for all $u_0 \in H$ if (H1) holds,

for all $u_0 \in D(A^{\gamma_2})$ if (H2) holds. Moreover

(i) $u \in C([0,T), H)$ if (H1) holds, $u \in C([0,T), D(A^{\gamma_2}))$ if (H2) holds;

(ii) $u(t) \in D(A)$ for all $t \in (0,T)$, $A(\cdot)u \in C^{\nu}((0,T), H)$ for some $\nu > 0$;

(iii) $u \in C^{1+\nu}((0,T), H)$.

In case of (H1) or in case of $\alpha_2 = 1$ in (H2) the solution is unique on $[0,T)$.

Remark 3: The reason for the difference in the required regularity of the initial condition in the two cases is the following: Since, in contrast to (F5), the conditions (F7), (F8) are only local with respect to u, we can't control singularities of $A^{\gamma_2}(\theta)u(t)$ as $t \downarrow 0$. In case of assumptions (H1) and $u_0 \in H$ we have $\|u(t)\|_m = O(t^{-\frac{1}{2}})$ as $t \downarrow 0$.

Proof: First we locally solve the Volterra integral equation related to (1.22):

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)F(s,u(s))ds \quad . \tag{1.23}$$

We show that (1.23) has a solution on some interval $[0,T_1]$ where T_1 is sufficiently small.

Let us assume (H1). To solve (1.23) we use an iteration method developed by Kato, Fujita [11], [8], and also Sobolevskii [25]. Therefore we only sketch the procedure. We define

$$S((0,T_1], V) = \{u \in C((0,T_1], V), \quad \sup_{(0,T_1]} t^{\frac{1}{2}}\|u(t)\|_m < \infty\} \tag{1.24}$$

and

$$u_1(t) = U(t,0)u_0$$

$$u_{n+1}(t) = u_1(t) + \int_0^t U(t,s)F(s,u_n(s))ds, \ n \geq 1 \ .$$

It is seen by induction that

(1.25) $$u_n \in C([0,T_1], H) \cap S((0,T_1], V)$$

for any $T_1 > 0$. Let $q_1 < 1$ be given. Since

(1.26) $$t^{1/2}\|A^{1/2}(\theta)U(t,0)u_0\| \to 0 \quad \text{for} \quad t \downarrow 0$$

(the proof of (1.26) is analogous to that of [8], p. 281, using (1.20)) it follows, again by induction, that for sufficiently small T_1

(1.27) $$\max_{[0,T_1]} \|u_n(t)\|, \quad \sup_{(0,T_1]} t^{1/2}\|u_n(t)\|_m \leq q_1, \ n \geq 1 \ .$$

(Observe that we assumed $\|g(t)\| = O(t^{-1+\varepsilon})$ for $t \downarrow 0$.)

Now we define

$$w_n = u_{n+1} - u_n, \ n \geq 1, \ w_0 = u_1 \ ,$$

and, if q_1 is sufficiently small, it is proved by induction that

(1.28) $$\max_{[0,T_1]} \|w_n(t)\|, \quad \sup_{(0,T_1]} t^{1/2}\|w_n(t)\|_m \leq q_2^n, \ n \geq 0 \ ,$$

for some $q_2 < 1$. This implies that

(1.29) $$u(t) = \lim_{n\to\infty} u_n(t) = \sum_{n=0}^{\infty} w_n(t)$$

converges in $C([0,T_1], H) \cap S((0,T_1], V)$ and that

(1.30) $$\lim_{n\to\infty} F(t,u_n(t)) = F(t,u(t)) \quad \text{for any} \quad t \in (0,T_1]$$

holds.

Observing the uniform estimate

$$\|U(t,s)F(s,u_n(s))\|_m \le c_{25}(T_1)(t-s)^{-\frac{1}{2}-\eta_0} s^{-1+\eta_0}, \quad n \ge 1 , \tag{1.31}$$

we can apply Lebesgue's theorem on dominated convergence and thus prove that $u \in C([0,T_1], H) \cap S((0,T_1], V)$ is actually a solution of the integral equation (1.23). We later show that u is actually a strict solution of (1.22).

Now we assume (H2). To solve (1.23) in this case we use the method of [12], [13]. Again, we only sketch the procedure. Define the mapping

$$V(u,v)(t) = U(t,0)u_0 + \int_0^t U(t,s)G(s,u(s),v(s))ds \tag{1.32}$$

on $X \times X$, where $X = C([0,T_1], D(A^{\gamma_2}))$. Then, in [12], [13] it is shown that V maps $X \times X$ into X and, for T_1 sufficiently small, it has the following properties (where $\|u\|_X = \max_{[0,T_1]} \|A^{\gamma_2}(\theta)u(t)\|$ for some fixed θ): There exists an $M_0 > 0$ such that

a) $\|V(u_1,v) - V(u_2,v)\|_X \le q_3\|u_1 - u_2\|_X$, $q_3 < 1$, $\|u_i\|_X$, $\|v\|_X \le M_0$,

b) $V(u,\cdot)$ is completely continuous as a mapping $X \to X$ for any fixed u with $\|u\|_X \le M_0$,

c) $\|V(0,v)\|_X(1 - q_3)^{-1} \le M_0$ for all $\|v\|_X \le M_0$.

(Observe that for property b) no compactness of $A^{-\gamma}(t)$ is needed.)

These three properties a) to c) imply that V has a fixed point u in X:

$$u = V(u,u) \quad \text{for some } u \in X, \ \|u\|_X \le M_0 . \tag{1.33}$$

This is a consequence of a more general fixed point theorem due to Darbo. In [12] we gave a simple proof which applies directly to this situation.

Next we show that the local solutions of (1.23) are indeed strict solutions of (1.22) on $(0,T_1]$.

Consider first the solution $u \in S((0,T_1], V)$. This implies that $u \in C([t_0,T_1], V)$ for any $0 < t_0 < T_1$. In view of (1.21) and (F3) we can prove

(1.34) $\quad u(t) \in D(A^{\gamma})$ and $\|A^{\gamma}(\theta)u(t)\| \le c_{26}(t_0,T_1)$ for all $t \in [t_0,T_1]$.

Indeed, in the first step we derive

(1.35) $\quad u(t) \in D(A^{\beta_1})$ and $\|A^{\beta_1}(\theta)u(t)\| \le c_{27}^{(1)}$ for $\frac{1}{2} < \beta_1 < 1 - \eta_0$.

This implies that $\|A^{-\eta_1}(\theta)F_0(t,u(t))\|$ is bounded, where

(1.36) $$1 - \rho(1 - \eta_0) < \eta_1 = 1 - \rho\beta_1 < \eta_0 \ .$$

Like in the first step this yields

(1.37) $\quad u(t) \in D(A^{\beta_2})$ and $\|A^{\beta_2}(\theta)u(t)\| \le c_{27}^{(2)}$ for $\beta_1 < \beta_2 < 1 - \eta_1$,

and thus a bound for $\|A^{-\eta_2}(\theta)F_0(t,u(t))\|$, where

(1.38) $$1 - \rho^2(1 - \eta_0) < \eta_2 = 1 - \rho\beta_2 < \eta_1 \ .$$

Since $\rho > 1$, in finitely many steps we reach $\eta_j = 0$ and β_{j+1} can be chosen an arbitrary number less than 1.

Now we apply (F1), (1.34), (1.16), and formula (2.9) in [23] (or Lemma 14.4 in [7], chap. 2) in order to derive that u is Hölder-continuous with some exponent $\tilde{\nu} > 0$:

(1.39) $$u \in C^{\tilde{\nu}}([t_0,T_1],\ D(A^{\gamma})) \ .$$

(The modulus of Hölder continuity depends on $t_0 > 0$, of course.)

The same holds for the solution $u \in X = C([0,T_1],\ D(A^{\gamma_2}))$ derived under hypothesis (H2). Assumptions (F6) or (F7) respectively together with the formula (2.25) in [23] (or (15.12) in [7], chap. 2) finally yield that u is actually a strict solution with the properties (i) to (iii) on the interval $(0,T_1]$.

Since $u(T_1) \in D(A)$ the local methods described above can a forteriori be repeated for the integral equation

$$\tilde{u}(t) = U(t,T_1)u(T_1) + \int_{T_1}^{t} U(t,s)F(s,\tilde{u}(s))ds \tag{1.40}$$

on some interval $[T_1,T_2]$. Clearly $\tilde{u}$ is an extension of the solution u of (1.23) to the interval $[0,T_2]$, and therefore we have a strict solution of (1.22) on $[0,T_2]$. If this process is repeated the sequence $T_1, T_2, \ldots$ might converge to some $T_0 < T$. If, however, $\limsup_{t\uparrow T_0} \|u(t)\|_m < \infty$, then the solution of the integral equation (1.23) on $[0,T_0)$ can be extended to $[0,T_0]$ by assigning an appropriate value $u(T_0) \in D(A^{\gamma_2})$. Indeed, if $\|u(t)\|_m \le c_{28}(t_0,T_0)$ on $[t_0,T_0)$, we consider

$$u(t) = U(t,t_0)u(t_0) + \int_{t_0}^{t} U(t,s)F(s,u(s))ds \quad . \tag{1.41}$$

Now, assumption (F4) (where we can replace γ_1 by $\gamma_2 \ge \gamma_1$) together with a generalized Gronwall lemma imply

$$\|A^{\gamma_2}(\theta)u(t)\| \le c_{29}(t_0,T_0) \quad \text{on} \quad [t_0,T_0) \quad . \tag{1.42}$$

This bound, assumption (F1) or (F7), and (1.16) show that for any sequence $t_n \uparrow T_0$ $\{u(t_n)\}$ is a Cauchy sequence in $D(A^{\gamma_2})$ and thus converges to a limit $u(T_0) \in D(A^{\gamma_2})$.

The global existence of a solution u on $(0,T)$ is thus proved if we can derive the following a-priori estimate for any strict solution:

$$\|u(t)\|_m \le c_{28}(t_0,T_0) \quad \text{on} \quad [t_0,T_0) \quad , \tag{1.43}$$

where $0 < t_0 < T_0 < T$.

Let u be a strict solution on $(0,T_0)$. Then by (A2) and (F0) we obtain for $t \in [t_0,T_0)$

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \mathrm{Re}(A(t)u,u) \le q\, c_1(T_0)\|u\|_m^2 + c_{19}(t_0,T_0)\|u\|^2 + \frac{1}{2}\|u\|^2 + \frac{1}{2}\|g\|^2$$

and

$$\frac{d}{dt}\|u\|^2 \le (2c_{19} + 1)\|u\|^2 + \|g\|^2 \quad .$$

These two estimates imply the energy inequality

$$(1.44)\quad \|u(t)\|^2 + \int_{t_0}^{t} \|u(s)\|_m^2\, ds \le c_{30}(t_0,T_0,\|u(t_0)\|,\ \|g\|_{L_2((t_0,T_0),H)}),\ t \in [t_0,T_0] .$$

By (F2) we get for the strict solution on $(0,T_0)$:

$$(1.45)\qquad \|A^{-\eta_0}(\theta)F_0(t,u(t))\| \le c_{31}(t_0,T_0)\|u(t)\|_m^{2(1-\eta_0)} ,\quad t \in [t_0,T_0) ,$$

(c_{31} depends also on $\|u(t_0)\|$ and g) and by (1.19), (1.20), and (1.21) $\|u(t)\|_m$ satisfies the integral inequality

$$(1.46)\qquad \|u(t)\|_m \le c_{32}\|u(t_0)\|_m + $$
$$+ c_{33}\int_{t_0}^{t} \{(t-s)^{-\frac{1}{2}-\eta_0}\|u(s)\|_m^{2(1-\eta_0)} + (t-s)^{-\frac{1}{2}}\|g(s)\|\}ds$$

for $t \in [t_0,T_0)$, where the constants c_{32}, c_{33} clearly depend on t_0,T_0, $\|u(t_0)\|$, and g. The two inequalities (1.44) and (1.46), together with the continuity of $\|u(t)\|_m$ on $[t_0,T_0)$, and the following Lemma 1.2 imply the boundedness (1.43) of $\|u(t)\|_m$ on $[t_0,T_0)$. Thus our main Theorem is proved.

Lemma 1.2. Let $\varphi \in C((0,T_0),\mathbb{R}_+)$ satisfy for all $t_0 \in (0,T_0)$

(i) $0 \le \varphi(t) \le \varphi(t_0) + \int_{t_0}^{t} \{(t-s)^{-\frac{1}{2}-\eta}\varphi(s)^{2(1-\eta)} + (t-s)^{-\frac{1}{2}}g(s)\}ds$ for $t \in [t_0,T_0)$, where

$0 \le \eta < \frac{1}{2}$ and $g \in L_{p,loc}((0,T_0),\mathbb{R}_+)$ for some $p > 2$;

(ii) $\varphi \in L_{2,loc}((0,T_0],\mathbb{R}_+)$.

Then there is a $t_1 \in (0,T_0)$ such that

$$\varphi(t) \le \max(1,2\varphi(t_1)) \quad \text{on} \quad [t_1,T_0) .$$

The proof of Lemma 1.2 will be given in the Appendix.

Section 2

Let $\Omega \subset \mathbb{R}^n$ be a domain with sufficiently regular boundary and let $A(t)$ be given by the differential operator

$$(2.1) \qquad A(t) \equiv \sum_{|\tilde{\alpha}| \le 2m} a^{\tilde{\alpha}}(t,x) D_{\tilde{\alpha}} \quad , \quad (t,x) \in [0,T) \times \Omega \quad ,$$

where $\tilde{\alpha} \in \mathbb{N}_0^n$ is a multi-index of length $|\tilde{\alpha}| = \sum_{i=1}^{n} \tilde{\alpha}_i$ and $D_{\tilde{\alpha}} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$ is a derivative of order $|\tilde{\alpha}|$ with respect to the space variable $x = (x_1,\dots,x_n)$. Since we want to admit systems of semilinear parabolic differential equations like (0.1), the $a^{\tilde{\alpha}}$ are in general $r \times r$ - matrices $(a_{ik}^{\tilde{\alpha}})$.

The ellipticity of (2.1) is expressed as follows:

$$(2.2) \qquad M(T_0)^{-1}|\xi|^{2m}|\eta|^2 \le \mathrm{Re} \sum_{i,k=1}^{r} \sum_{|\tilde{\alpha}|=2m} (-1)^m a_{ik}^{\tilde{\alpha}}(t,x)\xi^{\tilde{\alpha}}\eta_i\bar{\eta}_k \le M(T_0)|\xi|^{2m}|\eta|^2$$

for some $M(T_0) > 0$ and for all $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{C}^r$, and $(t,x) \in [0,T_0] \times \Omega$.

(In the terminology of [18] this is called "strong ellipticity".)

The coefficients a_{ik}^{α} are assumed to be smooth:

$$(2.3) \qquad a_{ik}^{\tilde{\alpha}}(t,\cdot) \in C^{2m}(\bar{\Omega}) \quad ,$$

$$(2.4) \qquad a_{ik}^{\tilde{\alpha}} \in C([0,T),C^m(\bar{\Omega})) \cap C^{\alpha}([0,T_0],C(\bar{\Omega}))$$

for all $T_0 < T$. In view of the conditions on η_0 (see (F2) and Theorem 2.1 below) we assume $\min(\frac{1}{2},\frac{n}{4m}) \le \alpha \le 1$.

Let the boundary conditions be given by

$$(2.5) \qquad B_j \equiv \sum_{|\tilde{\alpha}|<m_j} b^{\tilde{\alpha},j}(x) D_{\tilde{\alpha}} \quad , \quad 0 \le m_j \le m-1, \; j = 0,\dots,m-1 \quad ,$$

where

$$(2.6) \qquad b_{ik}^{\tilde{\alpha},j} \in C^{2m-m_j}(\partial\Omega) \quad .$$

We set $H = L_2(\Omega)$ and

$$V = \{u \mid u \in H^m(\Omega),\ B_j u = 0,\ j = 0,\dots,m-1\} \tag{2.7}$$

where the boundary conditions are fulfilled in the sense of the trace spaces (see e.g. [17]). (Actually, $L_2(\Omega)$ and $H^m(\Omega)$ are r-fold products of the usual spaces of scalar complex-valued functions. We won't denote this difference explicitely.) The space V endowed with the norm $\|\ \|_m$ of $H^m(\Omega)$ is a Hilbert space. Finally we have $\mathring{H}^m(\Omega) \subset V \subset H^m(\Omega)$, where $\mathring{H}^m(\Omega) = \{u \in H^m(\Omega),\ D_{\tilde{\alpha}} u = 0 \text{ on } \partial\Omega,\ |\tilde{\alpha}| \leq m-1\}$.

We don't want to give all possible boundary conditions $\{B_j\}$ which imply that assumptions (A1) to (A6) on $A(t)$ are fulfilled. First of all we assume that for any fixed $t \in [0,T)$ $A(t)$ together with the boundary conditions $\{B_j\}$ gives rise to a regular elliptic boundary value problem in the sense of [3], [4] (for $r = 1$ see also [17]).

If $r > 1$ this might be tedious to check. Therefore we confine ourselves to Dirichlet boundary conditions in this case: $V = \mathring{H}^m(\Omega)$. If we take $D(A) = H^{2m}(\Omega) \cap \mathring{H}^m(\Omega)$ all assumptions (A1) to (A6) on $A(t)$ are fulfilled.

Indeed, after integration by parts we get (A3) and (A2) is exactly Gårding's inequality (see [18]). Since the elliptic a-priori estimates are valid in this case (see [4] or [18]) Theorem 12.8 in [3] gives

$$\|u\|_{2m} \leq c_{34}(T_0)\,\|(A(t) + \lambda)u\| \quad , \quad u \in D(A),\ t \in [0,T_0] \quad , \tag{2.8}$$

where $\lambda > 0$ is sufficiently large.

The formal adjoint $A'(t)$ of $A(t)$ satisfies all conditions to assure inequality (2.8) with $A(t)$ replaced by $A'(t)$. That means that $A(t) + \lambda I$ as well as $A'(t) + \lambda I$ endowed with the domain $D(A)$ have closed ranges and thus by the closed range theorem they are surjective. Thus we have (A4), $D(A^*(t)) = D(A)$ for all t (which is (A5)$'$), and finally property (A6) follows from assumption (2.4) and the elliptic a-priori estimates which are uniform on $[0,T_0]$ because of the uniform bounds of the coefficients on $[0,T_0]$.

If $r = 1$ we can admit a larger class of boundary conditions such that the conditions on $A(t)$ are fulfilled. First of all, $A(t)$ together with the boundary conditions $\{B_j\}$ has to fit into the framework of the "variational boundary value problems" as they are

described in [17], chap. 2.9. We briefly explain what our conditions on $A(t)$ mean: (A3) says that $A(t)$ defines a continuous sesquilinear form $a(t; u,v)$ on V such that (for fixed t)

$$(2.9) \qquad a(t; u,v) = (A(t)u,v) \quad \text{for all} \quad v \in V,\ u \in D(A) \quad .$$

On the other hand, (A2) with $k = 0$ implies by the representation theorem due to Lax-Milgram that the form $a(t; \cdot , \cdot)$ defines a maximal operator $\tilde{A}(t)$ with some domain $D(\tilde{A}(t))$ which, due to assumption (A4), coincides with the given operator $A(t)(D(\tilde{A}(t)) = = D(A))$. Thus, for fixed t, $A(t)$ is regular or maximal accretive in the sense of [9], [16].

If the space V is characterized by the given boundary conditions as in (2.7) we have to assume that

$$(2.10) \qquad \{B_j\}_{j=0}^{m-1} \text{ is a "Dirichlet system" of order } m$$

in the sense of Def. 2.1 in [17], chap. 2.2. Then (2.9) is a consequence of Green's formula (see (2.19) in [17], chap. 2.2), which asserts (A3). The accretiveness (A2) (or coerciveness) of the form $a(t; \cdot , \cdot)$ on V is investigated by Agmon [1] (see also Theorem 9.3 in [17], chap. 2.9). We don't want to give all his conditions here.

Let $\tilde{A}(t)$ be the maximal operator defined by the form $a(t; \cdot , \cdot)$, where w.ℓ.o.g. $k = 0$ in the coerciveness inequality (A2). By the regularity results in [18] (observe especially the remarks on page 668; we assume that the conditions on $\{B_j\}$ required there are fulfilled) it follows that $D(\tilde{A}(t)) \subset H^{2m}(\Omega) \cap V$ for all t. Thus, if we endow $A(t)$ with the domain $D(A) = H^{2m}(\Omega) \cap V$, we get $A(t) = \tilde{A}(t)$ and $A(t)$ is regular accretive. Since the adjoint of $A(t)$ is defined by

$$(2.11) \qquad a(t; u,v) = (u, A^*(t)v) \quad \text{for all} \quad u \in V,\ v \in D(A^*(t)) \quad ,$$

we can apply the same regularity argument in order to derive that $D(A^*(t)) = D(A) = H^{2m}(\Omega) \cap V$ for all t.
Thus (A1) to (A5)' are fulfilled and (A6) follows by the elliptic a-priori estimates established in [18].

Remark 4: The problem of coerciveness of a sesquilinear form on V was considered by Agmon when V is defined only by p boundary conditions B_j, $j = 0,\dots,p-1$, where $0 \le p \le m$. If $p < m$, then all functions in the domain of $\tilde{A}(t)$, where $\tilde{A}(t)$ is defined by $a(t,\cdot,\cdot)$ via (2.9), fulfill in addition $m - p$ so called "natural boundary conditions" N_j, $j = p,\dots,m-1$, of some order between m and $2m - 1$. These natural boundary conditions depend on the form $a(t,\cdot,\cdot)$ and B_j, $j = 0,\dots,p-1$, and thus, in general are not time independent if $a(t,\cdot,\cdot)$ is not.

If, however, $a(\cdot,\cdot)$ is time independent, all these "variational boundary value problems" defined by a coercive form $a(\cdot,\cdot)$ are admitted. Again by the regularity proof in [18] we get for the domain of the operator $\tilde{A} = A$

$$D(A) = H^{2m}(\Omega) \cap V \cap \{u \in H^{2m}(\Omega),\ N_j u = 0,\ j = p,\dots m-1\}$$

and for the domain of A^* defined by (2.11)

$$D(A^*) = H^{2m}(\Omega) \cap V \cap \{u \in H^{2m}(\Omega),\ N_j^* u = 0,\ j = p,\dots,m-1\} ,$$

where N_j, N_j^* are not necessarily the same natural boundary conditions.

If we choose $Y = H^{2m}(\Omega)$, we have

$$V \subset H^m(\Omega) = [Y,H]_{1/2},\ D(A),\ D(A^*) \subset Y ,$$

so that (A5) is fulfilled in this case.

As indicated already in Section 1, Lions [16] showed under the assumption of regular accretiveness that

$$D(A^\gamma) = [D(A),\ H]_{1-\gamma} \quad , \quad 0 \le \gamma \le 1 , \tag{2.12}$$

where $[D(A),\ H]_{1-\gamma}$ is the interpolation space between $D(A)$ and H of order $1 - \gamma$. (If $A(t)$ is self-adjoint (2.12) can be found in [17], chap. 1.2. If $A(t)$ is not self-adjoint, (2.12) basically results from Kato's generalization of the Heinz inequality [10].)

In view of the continuous embeddings $D(A)$, $D(A^*) \subset H^{2m}(\Omega)$, we get by the interpolation theorem (Theorem 5.1 in [17], chap. 1.5):

(2.13) $$D(A^{\gamma}) \subset H^s(\Omega),\ D(A^{*\gamma}(\theta)) \subset H^s(\Omega), \quad \text{for} \quad s = 2m\gamma \ .$$

(For the definition of the spaces $H^s(\Omega)$ for real s as interpolation spaces, see [17], chap. 1.9.)

Furthermore, for the spaces $H^s(\Omega)$ we have the analogous embedding theorems as for $H^m(\Omega)$ with integer m (see e.g. [21]):

(2.14) $$H^s(\Omega) \subset L_p(\Omega) \quad \text{for} \quad p \leq \frac{2n}{n-2s}\ ,\ s < \frac{n}{2}\ ,$$

(2.15) $$H^s(\Omega) \subset C^{\mu}(\bar{\Omega}) \quad \text{for} \quad s \geq \frac{n}{2} + \mu,\ 0 < \mu < 1 \ .$$

If $s = \frac{n}{2}$, p *in (2.14) can be chosen arbitrarily. The norms in* $H^s(\Omega)$ *will be denoted by* $\|\ \|_s$.

After these remarks we are ready to give concrete realizations of the nonlinearity F satisfying the conditions (Fi).

In general F is given by

$$F(t,u) = (F^1(t,u),\dots,F^r(t,u)) \ ,\quad F(t,u) = F_0(t,u) + g(t) \ ,$$

$$g = (g^1,\dots,g^r) \ ,\quad g^k = g^k(t,x) \ ,\quad (t,x) \in (0,T)\times\Omega,\ k = 1,\dots,r \ ,$$

(2.16) $$F_0^k(t,u) = f^k(t,x,D_{\tilde{\gamma}_1} u^{i_1},\dots,D_{\tilde{\gamma}_\ell} u^{i_\ell}) \ ,\quad 1 \leq i_\kappa \leq r,\ |\tilde{\gamma}_\kappa| \leq m \ ,$$

$$\text{if} \quad u = (u^1,\dots,u^r) \ .$$

Before stating and proving our main Theorem we remark that without loss of generality we can restrict ourselves to the case $r = 1$.

<u>Theorem 2.1</u>. Let the measurable function

$$f : (0,T)\times\Omega\times\mathbb{C}^{\ell} \to \mathbb{C} \quad \text{satisfy for} \quad z \in \mathbb{C}^{\ell}$$

(i) $$|f(t,s,z_1,\dots,z_\ell)| \leq \mu_0(t) \sum_{\kappa=1}^{\ell} |z_\kappa|^{r_\kappa},\ 1 \leq r_\kappa \leq \frac{4m+n}{|\tilde{\gamma}_\kappa|+n} = R(|\tilde{\gamma}_\kappa|)$$

for all $(t,x) \in (0,T)\times\Omega$, where $\mu_0 \in C([0,T), \mathbb{R}_+)$.

Then $F_0(t,u)$ given by (2.16) fulfills:

1) Condition (F1) with $\gamma = \frac{2m+n}{4m+n}$, $\psi_1(y) = \sum_{\kappa=1}^{\ell} y^{R_\kappa}$, and some constant depending on Ω, $A(t)$, $\mu_0(t)$ for $t \in [t_0,T_0]$ ($R_\kappa = R(|\gamma_\kappa|)$ by definition).

2) Condition (F2) with $1 - \frac{1}{2}\gamma^{-1} = \eta_0 < \min(\frac{1}{2}, \frac{n}{4m})$, $\psi_2(y) = \sum_{\kappa=1}^{\ell} y^{R_\kappa - 2(1-\eta_0)}$, $\mu_1(t) = c_{35}\mu_0(t)$,

3) Condition (F3) with $\eta = \eta(\beta) = 1 - \beta\gamma^{-1}$, $\psi_3 = \psi_1$ ($\rho = \gamma^{-1} > 1$),

4) Condition (F4) with $\gamma_1 \geq \frac{1}{2}(3 - \gamma^{-1})$, $\psi_4(y) = \sum_{\kappa=1}^{\ell} y^{R_\kappa - 1}$, $\mu_2(t) = c_{36}\mu_0(t)$.

If moreover

(ii) $|f(t_1,x,z) - f(t_2,x,z)| \leq c_{37}(t_0,T_0)h_{\tilde M}(x)|t_1-t_2|^{\alpha_1}$

for all $t_0 > 0$, $t_i \in [t_0,T_0]$, $x \in \Omega$, $|z| \leq \tilde M$, $h_{\tilde M} \in L_2(\Omega)$, $0 < \alpha_1 \leq 1$;

(iii) f is differentiable with respect to z and

$$\left|\frac{\partial}{\partial z_i} f(t,x,z)\right| \leq \mu_0(t) \sum_{\kappa=1}^{\ell} |z_\kappa|^{r_\kappa - 1}, \quad 1 \leq r_\kappa \leq R(|\tilde\gamma_\kappa|), \quad i = 1,\dots,\ell,$$

then $F_0(t,u)$ fulfills:

5) Condition (F5) with $\mu_3(t) = c_{38}\mu_0(t)$,

6) Condition (F6) with $\psi_5 = \psi_4$.

Finally, let instead of (iii) only the following local estimates hold ($z = (z^1,z^2)$, $z^1 = (z_1,\dots,z_{\ell_1})$, $z^2 = (z_{\ell_1+1},\dots,z_\ell)$):

(iv) $|f(t_1,x,z) - f(t_2,x,z)| \leq h_{T_0,\tilde M}(x)|t_1-t_2|^{\alpha_1}$

for $t_1,t_1 \in [0,T_0]$, $x \in \Omega$, $|z| \leq \tilde M$, $h_{T_0,\tilde M} \in L_2(\Omega)$, $0 < \alpha_1 \leq 1$;

(v) $|f(t,x_1,z) - f(t,x_2,z)| \leq |\tilde{f}_{T_0,\tilde{M}}(x_1) - \tilde{f}_{T_0,M}(x_2)|$

for $t \in [0,T_0]$, $x_i \in \Omega$, $|z| \leq \tilde{M}$, $\tilde{f}_{T_0,\tilde{M}} \in L_2(\Omega)$;

(vi) $|f(t,x,z^1,z^2) - f(t,x,\tilde{z}^1,z^2)| \leq c_{39}(T_0,\tilde{M})|z^1 - \tilde{z}^1|$

for $t \in [0,T_0]$, $x \in \Omega$, $|z^1|$, $|\tilde{z}^1|$, $|z^2| \leq \tilde{M}$;

(vii) $|f(t,x,z^1,z^2) - f(t,x,z^1,\tilde{z}^2)| \leq \sum_{\kappa=\ell_1+1}^{\ell} |g^{\kappa}_{T_0,\tilde{M}}(x)|\ |z_\kappa - \tilde{z}_\kappa|^{\rho_\kappa}$

for $t \in [0,T_0]$, $x \in \Omega$, $|z^1|$, $|z^2|$, $|\tilde{z}^2| \leq \tilde{M}$, $0 < \rho_\kappa < 1$, $g^{\kappa}_{T_0,\tilde{M}} \in L_{2/(1-\rho_\kappa)}(\Omega)$.

Assume in addition $n < 4m$. Then $F_0(t,u)$ fulfills:

7) Condition (F7) with $\frac{n}{4m} < \gamma_2 < 1$ and $\alpha_2 = \min\{\rho_i\}$,

8) Condition (F8).

Remark 5: Assumptions (ii), (iv) to (vii) are formulated for unbounded domains Ω.

Remark 6: The restriction on the dimension $n < 4m$ is caused by the local character of the conditions (iv) to (vii). We have to assure that $\|A^{\gamma_2}(\theta)u(t)\| \leq M$ implies a pointwise bound of the function $|u(t,x)|$, namely $|u(t,x)| \leq \tilde{M}$.

Proof: We prove (F2) to (F6) under the assumption $r_\kappa = R_\kappa = R(|\tilde{\gamma}_\kappa|)$. If $1 \leq r_\kappa < R_\kappa$ the estimates hold a forteriori.

1) Estimate (F1) is simply a consequence of the continuous embeddings (2.13), (2.14).

2) Let $v \in L_2(\Omega)$ be arbitrary. Then

$$|(v, A^{-n_0}(\theta)F_0(t,u))| = |(A^{*-n_0}(\theta)v, F_0(t,u))|$$

(here we used $(A^{-\gamma})^* = A^{*-\gamma}$, which can be seen by definition (1.19) and $(e^{-\tau A})^* = e^{-\tau A^*}$)

$$\leq \|A^{*^{-\eta_0}}(\theta)v\|_{L_p(\Omega)} \|F_0(t,u)\|_{L_q(\Omega)} \ , \quad \frac{1}{p}+\frac{1}{q}=1 \ ,$$

$$\leq c_{40}(T_0)\|v\|\mu_0(t)\sum_{\kappa=1}^{\ell}\|D_{\tilde{\gamma}_\kappa}u\|^{R_\kappa}_{L_{qR_\kappa}(\Omega)} \ , \quad p=\frac{2n}{n-4m\eta_0} \ ,$$

$$\leq c_{41}(T_0)\|v\|\mu_0(t)\sum_{\kappa=1}^{\ell}\|u\|^{R_\kappa}_{s_\kappa} \ , \quad qR_\kappa=\frac{2n}{n-2(s_\kappa-|\tilde{\gamma}_\kappa|)} \ ,$$

where $s_\kappa = m\frac{2|\tilde{\gamma}_\kappa|+n}{2m+n}$. (We also used that $D_{\tilde{\gamma}}$ is a bounded linear operator from $H^s(\Omega)$ into $H^{s-|\tilde{\gamma}|}(\Omega)$ for any real $s \geq |\tilde{\gamma}|$; see [17] Theorem 9.7, chap. 1.9.) Combining these conditions on the conjugate exponents p and q and taking into account the definitions of R_κ and s_κ we get

$$2(1-\eta_0) = R_\kappa\frac{s_\kappa}{m} = \frac{4m+n}{2m+n} \ . \tag{2.17}$$

Now we make use of the interpolation inequality for the norms in $H^s(\Omega)$ (see [17], chap. 1.9):

$$\|u\|_s \leq c_{42}(s,m)\|u\|_m^{s/m}\|u\|^{1-(s/m)}, \quad 0\leq s\leq m \ , \tag{2.18}$$

which finally yields

$$|(v, A^{-\eta_0}(\theta)F_0(t,u))| \leq c_{43}(T_0)\|v\|\mu_0(t)\|u\|_m^{2(1-\eta_0)}\sum_{\kappa=1}^{\ell}\|u\|^{R_\kappa-2(1-\eta_0)} \ .$$

Since v is arbitrary (F2) is proved.

3) The same argument proves also (F3) (choose $s_\kappa = 2m\beta\frac{2|\tilde{\gamma}_\kappa|+n}{2m+n}$) .

4) By assumption (i) we get

$$\|F_0(t,u)\| \le \mu_0(t) \sum_{\kappa=1}^{\ell} \|D_{\tilde{\gamma}_\kappa} u\|_{L_{2p}(\Omega)} \|D_{\tilde{\gamma}_\kappa} u\|^{R_\kappa - 1}_{L_{2(R_\kappa-1)q}(\Omega)} , \quad \frac{1}{p} + \frac{1}{q} = 1 ,$$

$$\le c_{44}(T_0)\mu_0(t) \|A^{\gamma_1}(\theta)u\| \sum_{\kappa=1}^{\ell} \|u\|_m^{R_\kappa - 1} , \quad \text{if}$$

(2.19) $$2p \le \frac{2n}{n-2(2m\gamma_1 - |\tilde{\gamma}_\kappa|)} , \quad 2(R_\kappa - 1)q \le \frac{2n}{n-2(m-|\tilde{\gamma}_\kappa|)} .$$

(If one of the denominators in (2.9) is less than or equal to zero then the conjugate exponents can be chosen arbitrarily.)

Relations (2.19) yield the condition

(2.20) $$\gamma_1 \ge \frac{1}{2}(3-R_\kappa) , \quad \kappa = 1,\dots,\ell ,$$

which is fulfilled by $\gamma_1 = \frac{1}{2}(3-\gamma^{-1})$. Observe that $\gamma_1 < 1$.

The proofs of conditions (F5) and (F6) are analogous to those presented here so that we omit them.

Finally for the proof of (F6) and (F7) we refer to [12], Satz 5.5. Thus Theorem 2.1 is proved.

Condition (F0) on $F(t,u)$ is not expressed in terms of the function f by the same reasons we gave in the introduction. This *"sign-condition" (F0) has only to be fulfilled for* u in the domain $D(A)$ which possibly allows integrations by parts. Typical examples are given by

$$F(u) = \sum_{\kappa=1}^{n} u^{r_\kappa} \frac{\partial u}{\partial x_\kappa} , \quad 1 \le r_\kappa \le \frac{4m-2}{n} ,$$

when $D(A) = H^{2m}(\Omega) \cap \mathring{H}^m(\Omega)$ or

$$F(u) = -\sum_{\kappa=1}^{\ell} u|D_{\tilde{\gamma}_\kappa} u| , \quad |\tilde{\gamma}_\kappa| \le 2m - \frac{n}{2} ,$$

for any $D(A) = H^{2m}(\Omega) \cap V$. In case of the Navier-Stokes nonlinearity the fact that

$\operatorname{div} u = 0$ for $u \in D(A)$ is the reason why $((u\cdot\nabla)u,u) = 0$ for all u vanishing on the boundary $\partial\Omega$.

Application of Theorem 1.1 yields global strict solutions of the evolution equation (0.2) in $H = L_2(\Omega)$ where $A(t)$ and F are given by the differential operators described in this Section. (For a characterization of the initial value $u_0 \in D(A^{\gamma_2})$ see (2.27) below.) As mentioned in the introduction, however, a classical solution of (0.1) is desired.

We finish with some remarks about how to prove that a strict solution of (0.2) is actually a classical solution of (0.1). Let us assume for simplicity that the data of problem (0.1) are smooth (i.e. the coefficients of $A(t)$, B_j, the boundary $\partial\Omega$, and f). Let us further assume that, for fixed t, $A(t)$ together with the boundary conditions gives rise to a regular elliptic boundary value problem in any space $L_p(\Omega)$, $p > 2$, and also in the Hölder spaces $C^{\mu}(\bar{\Omega})$ for $0 < \mu < 1$.

By a resolvent estimate due to Agmon [2] $-A(t)$ generates an analytic semigroup in any $L_p(\Omega)$ and, moreover, the family $A(t)$ generates a fundamental solution $U_p(t,s)$ in $L_p(\Omega)$ with the properties listed in Section 1 (for the details see [7]). The operator $A(t)$ considered in $L_p(\Omega)$ will be denoted by $A_p(t)$.

Let u be a solution in $C^{\nu}((0,T), D(A)) \subset C^{\nu}((0,T), H^{2m}(\Omega))$. By the assumptions on f given in Theorem 2.1 we have

$$F_0(\cdot,u) \in C^{\nu_1}((0,T), L_{p_1}(\Omega)) \quad \text{for some } \nu_1 > 0 ,$$

where $p_1 = \frac{2n(2m+n)}{(4m+n)(n-2m)} > 2$. (If $2m = n$, p_1 can be chosen arbitrarily, if $2m > n$ we go immediately to the steps described below.) Considering F as an inhomogeneity, the results in [23], [7] show that u is actually in $C^{\nu}((0,T), D(A_{p_1})) \subset C^{\nu}((0,T), W^{2m}_{p_1}(\Omega))$. Repeating this argument we get sequentially

$$F_0(\cdot,u) \in C^{\nu_1}((0,T), L_{p_k}(\Omega)) ,$$

where
$$p_{k+1} = \frac{p_k n(2m+n)}{(4m+n)(n-p_k n)} > p_k , \quad k = 1,2,\ldots .$$

Obviously we have $p_j m > n$ for some j. An embedding theorem due to Morrey, analogous to (2.15), yields

$$\text{(2.21)}\qquad \begin{aligned} & u \in C^{\nu}((0,T),\, W^{2m}_{p_j}(\Omega)) \subset C^{\nu}((0,T),\, C^{\mu}(\bar{\Omega})) \;, \\ & F_0(\cdot,u) \in C^{\nu_1}((0,T),\, C^{\mu}(\bar{\Omega})) \quad \text{for some } \mu > 0 \;. \end{aligned}$$

We tacitly assumed that $g \in C^{\nu_1}((0,T),\, L_p(\Omega))$ for all $p > 2$. Let us further assume that $g \in C^{\nu_1}((0,T),\, C^{\mu}(\bar{\Omega}))$. Then the results in [23] (see formulas (2.7), (1.70), (1.71)) yield:

$\frac{du}{dt}$ exists in the topology of $D(A_p^{\gamma_0}(\theta))$ with $\gamma_0 < \min(\nu_1,\alpha)$ and

$\frac{du}{dt} \in C((0,T),\, D(A_p^{\gamma_0}(\theta)))$ for some fixed θ and all $p > 0$.

The embeddings (see e.g. [12], [21])

$$\text{(2.22)}\qquad D(A_p^{\gamma_0}(\theta)) \subset W_p^s(\Omega) \subset C^{\mu}(\bar{\Omega}) \quad \text{for} \quad 2m\gamma_0 > s \geq \frac{n}{p} + \mu \;,$$

which are certainly true for sufficiently large p and some small μ, finally gives

$$\text{(2.23)}\qquad A(t)u(t) = -\frac{du}{dt}u(t) + F(t,u(t)) \in C^{\mu}(\bar{\Omega}) \quad \text{for fixed } t \;,$$

the right hand side being continuous in $C^{\mu}(\bar{\Omega})$ with respect to t. Furthermore (2.21) implies that the boundary conditions are fulfilled by $u(t)$ in the classical sense. Now, the Schauder estimates for elliptic boundary value problems [3] applied to (2.23) prove that $u(t) \in C^{2m+\mu}(\bar{\Omega})$ which completes the proof that u is a classical solution of (0.1).

(In (2.22) we introduced the spaces $W_p^s(\Omega)$ for real s which, similar to the case $p = 2$, are defined as interpolation spaces between $W_p^{\ell}(\Omega)$ and $L_p(\Omega)$, $0 < s < \ell$. The definition is independent of the integer ℓ (see [21]). For integer s they coincide with the usual Sobolev spaces, of course. The same holds for the spaces $\mathring{W}_p^s(\Omega)$, $s \neq \ell + \frac{1}{p}$.)

An alternate procedure to proof regularity is the following: Once we know that $F(\cdot,u) \in C^{\nu_1}((0,T),\, L_p(\Omega))$ for some $p > \frac{n}{m}$, we can show by Morrey's embedding theorem that

$$F(\cdot,u) \in C^{\nu_1}((0,T),\ C_*^{\mu}(\overline{\Omega})), \text{ where}$$

$$C_*^{\mu}(\overline{\Omega}) = \{u \in C^{\mu}(\overline{\Omega}),\ \lim_{K\to\infty} \|u\|_{C^{\mu}(\overline{\Omega}_K)} = 0\},\ \Omega_K = \Omega \cap \{x \in \mathbb{R}^n,\ |x| > K\}\ .$$

Now we apply the results in [13], especially Lemmas 6.7 and 6.8, in order to derive that u is actually a solution of the evolution equation (0.2) in $C_*^{\mu}(\overline{\Omega})$ which means that u is a forteriori a classical solution of (0.1).

Possible generalizations

We briefly indicate possible generalizations, especially of Theorem 0.1.

To this purpose we introduce the abbreviations $\|\ \|_{p,s}$ for the norms in the spaces $W_p^s(\Omega)$ for real s. Clearly $\|\ \|_{p,0} = \|\ \|_{L_p(\Omega)}$.

We assume again that $A_p(t)$ with some time independent domain $\mathring{W}_p^{2m}(\Omega) \subset D(A_p) \subset W_p^{2m}(\Omega)$ generates a fundamental solution $U_p(t,s)$ with the properties listed in Section 1. Let F be given by (2.16) where now derivatives up to order $2m-1$ are admitted: $|\tilde{\gamma}_\kappa| \le 2m-1$.

Theorem 2.2. Assume that $f^j : (0,T) \times \Omega \times \mathbb{C}^{\ell} \to \mathbb{C}$ satisfies

(i) $$|f^j(t,x,z_1,\dots,z_\ell)| \le \mu_0(t) \sum_{\kappa=1}^{\ell} |z_\kappa|^{r_\kappa},\ 1 \le r_\kappa < \frac{2m-s+(n/p)}{|\tilde{\gamma}_\kappa|-s+(n/p)} = R(|\tilde{\gamma}_\kappa|,\ s)\ ,$$

for some $0 \le s \le 2m$ (if $|\tilde{\gamma}_\kappa| + (n/p) \le s$, r_κ is arbitrary), $j = 1,\dots,r$;

(ii) $$\left|\frac{\partial}{\partial z_i} f^j(t,x,z_1,\dots,z_\ell)\right| \le \mu_0(t) \sum_{\kappa=1}^{\ell} |z_\kappa|^{r_\kappa - 1},\ i = 1,\dots,\ell\ ;$$

for some $\mu_0 \in C([0,T),\ \mathbb{R}_+)$;

(iii) $$|f^j(t_1,x,z) - f^j(t_2,x,z)| \le c_{45}(t_0,T_0) h_{\tilde{M}}(x) |t_1-t_2|^{\alpha_1}$$

for all $t_0 > 0$, $t_i \in [t_0,T_0]$, $x \in \Omega$, $|z| \le \tilde{M}$, $h_{\tilde{M}} \in L_p(\Omega)$, $0 < \alpha_1 \le 1$;

(iv) $$g \in C^{\alpha_1}((0,T),\ L_p(\Omega)),\ \|g(t)\|_{p,0} = O(t^{-1+\gamma_3}),\ \gamma_3 > \frac{s}{2m}\ .$$

Then the initial value problem

$$\frac{du}{dt} + A_p(t)u = F(t,u) \tag{2.24}$$

$$u(0) = u_0$$

possesses for any $u_0 \in D(A_p^{\gamma_3}(\theta))$ a unique local strict solution in $E = L_p(\Omega)$ (θ is some fixed time in $[0,T)$).

Assume moreover that

$$\|u(t)\|_{p,s} \le c_{46}(t_0,T_0) \quad , \quad t \in [t_0,T_0] \subset (0,T) \quad . \tag{2.25}$$

Then the strict solution of (2.24) exists globally on $(0,T)$.

Remark 7: If $s = 0$, $u_0 \in L_p(\Omega)$ is possible ($\gamma_3 = 0$). For the characterization of the initial condition in the case $s > 0$ we can use the continuous embedding (see [13])

$$\mathring{W}_p^{\sigma}(\Omega) \subset D(A_p^{\gamma_3}(\theta)) \subset W_p^s(\Omega) \quad \text{for} \quad \sigma > 2m\gamma_3 > s \quad . \tag{2.26}$$

If $p = 2$, in view of $\mathring{H}^{2m}(\Omega) \subset D(A)$ and (2.12), this can be replaced by

$$\mathring{H}^s(\Omega) \subset D(A^{\gamma_3}) \subset H^s(\Omega) \quad \text{for} \quad s = 2m\gamma_3 \quad . \tag{2.27}$$

Furthermore, instead of the global assumption (ii), we could also assume local conditions analogous to (iv) to (vii) in Theorem 2.1. The initial condition must be in $D(A_p^{\gamma_3}(\theta))$ in this case, where $\frac{n}{2mp} < \gamma_3 < 1$. Then the same local and global result of Theorem 2.2 is true. The proof is completely analogous to that of Theorem 1.1 (see also [13]).

<u>Proof</u>: Using the interpolation inequality

$$\|u\|_{p,\sigma} \le c_{47}\|u\|_{p,t}^{(\sigma-s)/(t-s)} \; \|u\|_{p,s}^{(t-\sigma)/(t-s)} \quad \text{for} \quad 0 \le s < \sigma < t \quad , \tag{2.28}$$

which follows from the definition as interpolation spaces, using the embedding (2.22), we obtain by assumption (i) for $t \in (0,T_0]$

$$\|F_0(t,u)\|_{p,0} \le c_{48}(T_0)\,\|A_p^{\gamma_4}(\theta)u\|_{p,0} \sum_{\kappa=1}^{\ell} \|u\|_{p,s}^{r_\kappa - 1} \quad \text{for some} \quad \gamma_4 < 1 \quad . \tag{2.29}$$

(If $r_\kappa = R(|\tilde{\gamma}_\kappa|, s)$, then $\gamma_4 = 1$. Therefore the limiting growth rates cannot be permitted.)

Now, the proof uses the iteration method developed by Kato, Fujita, and Sobolevskii: Define

$$S((0,T_1], D(A_p^{\gamma_4}(\theta))) = \{u \in C((0,T_1], D(A_p^{\gamma_4}(\theta))), \sup_{(0,T_1]} t^{\gamma_4-\gamma_3'} \|A_p^{\gamma_4}(\theta)u(t)\|_{p,0} < \infty\}$$

for some $\gamma_3 > \gamma_3' > \frac{s}{2m}$, and

$$u_1(t) = U_p(t,0)u_0 \quad ,$$

$$u_{n+1}(t) = u_1(t) + \int_0^t U_p(t,s)F(s,u_n(s))ds, \quad n = 1,2,\dots \quad .$$

Using (1.14) (which is also valid in $L_p(\Omega)$), the relation

$$\text{(2.30)} \qquad \|A_p^{\gamma'}(t_1)A_p^{-\gamma}(t_2)\| \le c_{49}(T_0) \quad \text{for} \quad 0 \le \gamma' < \gamma, \ t_i \in [0,T_0] \quad ,$$

and the estimate (2.29) yield by induction

$$u_n \in C([0,T_1], D(A_p^{\gamma_3'}(\theta)) \cap S((0,T_1], D(A_p^{\gamma_4}(\theta))) \quad .$$

The further proof follows exactly the lines as described in Section 1, since we have by assumption (ii)

$$\text{(2.31)} \quad \|F_0(t,u_1) - F_0(t,u_2)\|_{p,0} \le c_{50}(T_0)\|A_p^{\gamma_4}(\theta)(u_1-u_2)\|_{p,0} \sum_{\kappa=1}^{\ell} (\|u_1\|_{p,s}^{r_\kappa-1} + \|u_2\|_{p,s}^{r_\kappa-1}) \quad .$$

The global result follows from (2.29) which yields for $\|A_p^{\gamma_4}(\theta)u(t)\|_{p,0}$ a linear integral inequality and thus a bound on any interval $[t_0,T_0] \subset (0,T)$. Clearly, if the data are smooth enough, this global solution is a classical solution of (0.1).

Theorem 2.2 applies to the situation in [20] where $p = 2$ and $s = m$. However, our proof if considerably simpler since no a-priori estimates of Solonnikov are used. Furthermore, as long as the function f is locally Lipschitz-continuous (see assumption (vi) of Theorem 2.1 with $z^1 = z$) in contrast to [20] no compactness is needed at all.

Corollary 2.3. Assume that Ω is bounded and that, instead of (2.25), the a-priori bound

$$\|u(t)\|_{C^{\mu}(\Omega)} \leq c_{51}(t_0,T_0) \quad , \quad t \in [t_0,T_0] \subset (0,T) \quad , \tag{2.32}$$

is valid. Then in assumption (i) the growth rate

$$r_{\kappa} < \frac{2m-\mu}{|\tilde{\gamma}_{\kappa}|-\mu} \tag{2.33}$$

will suffice to assure the global existence of (2.24) in $E = L_p(\Omega)$, where $p < \infty$ is sufficiently large.

Proof. For bounded domains we have the continuous embedding

$$C^{\mu}(\bar{\Omega}) \subset W_p^s(\Omega) \quad \text{for} \quad \mu \geq s \quad \text{and any} \quad 1 < p < \infty \quad . \tag{2.34}$$

This follows by interpolation since $C^1(\bar{\Omega}) \subset W_p^1(\Omega)$ and $C^0(\bar{\Omega}) \subset W_p^0(\Omega) = L_p(\Omega)$ (see [13], p. 140).

This Corollary is the main result in a recent paper of W. von Wahl: "Über das Grösstmögliche Wachstum der Nichtlinearität bei Semilinearen Parabolischen Gleichungen Beliebiger Ordnung", J. Funct. Anal. 27 (1978), 118-135. (He also admits, however, $u_0 \in C^{\mu}(\bar{\Omega})$ as initial condition for the corresponding Volterra integral equation when μ is sufficiently small.)

It would be interesting to know, for instance by a counter-example, whether in (2.33) equality is allowed or not. If not, then the growth rate (2.33) could be considered as best possible. The same question should be asked for all growth rates given in our paper, of course. In case of Theorem 2.1, where equality is allowed, the growth rates $r_{\kappa} = R_{\kappa}$ seem to be the best possible, in general. In case of Theorem 2.2, however, we leave it open. It will be rather tedious to construct counter-examples.

It would also be interesting to know, when a-priori bounds (2.25), (2.32) can be obtained. For (2.25) with $p = 2$, $s = m$ one answer is given in [19], [20]. An alternate method would be using sign-conditions in (0.13) in order to obtain an a-priori bound for

$\|u_t\|$ and thus again (2.25) with $p = 2$, $s = m$. Then $A(t)$ need not be self-adjoint and also derivatives in the nonlinearity could be allowed.

As for (2.32), only the case $\mu = 0$ for second order equations (maximum principle) covers a reasonably large class of problems. Then the global existence result is included in Theorem 0.1, of course. It should be mentioned, however, that for $\mu = 0$ and second order equations the growth rate (2.33) is not the best possible since quadratic growth in the first derivatives can be admitted (see [15]). For $\mu > 0$ the a-priori bound (2.32) is more or less of academic interest only.

Appendix

Proof of Lemma 1.1: Let $u \neq 0$, $u \in D(B)$, and $v = \|u\|^{-1}u$. Then $\|v\| = 1$ and

$$\operatorname{dist}(\lambda,S(B)) \leq |(Bv,v) - \lambda| = |((B-\lambda)v,v)| \leq \|(B-\lambda)v\|$$
$$= \|u\|^{-1}\|(B-\lambda)u\| .$$

So $B-\lambda I$ is injective and the range of $B-\lambda I$ is closed provided $\operatorname{dist}(\lambda,S(B)) > 0$. If moreover $\lambda \in P(B)$ then

$$\|(B-\lambda I)^{-1}\|^{-1} \geq \operatorname{dist}(\lambda,S(B)) \quad . \tag{3.1}$$

Consider the set $P(B) \cap P_0 \neq \emptyset$. It is clearly relatively open in P_0. It is also relatively closed in P_0. Indeed, if $\lambda_n \in P(B) \cap P_0$ and $\lambda_n \to \lambda \in P_0$, then $\operatorname{dist}(\lambda,S(B)) > 0$. For large enough n we have $\operatorname{dist}(\lambda_n,S(B)) > \frac{1}{2}\operatorname{dist}(\lambda,S(B))$ and, again for large enough n, $|\lambda-\lambda_n| < \operatorname{dist}(\lambda_n,S(B))$. This implies by (3.1) that λ is in a ball of radius less than $\|(B-\lambda_n I)^{-1}\|^{-1}$ around λ_n. Therefore $\lambda \in P(B)$. By the connectedness of P_0 it follows that $P(A) \cap P_0 = P_0$ or $P_0 \subset P(B)$. Finally

$$\|(B-\lambda I)^{-1}\| \leq \operatorname{dist}(\lambda,S(B))^{-1} \quad \text{for all} \quad \lambda \in P_0$$

follows from (3.1).

Proof of Lemma 1.2: Because of assumption (ii) for all $\varepsilon_1, \varepsilon_2 > 0$ there exists a $t_1 = t_1(\varepsilon_1,\varepsilon_2) \in (T_0-\varepsilon_2,T_0)$ such that

$$\varphi(t_1)^2 \leq \frac{\varepsilon_1}{T_0-t_1} \quad . \tag{3.2}$$

Choose

ε_1 such that $\dfrac{8}{1-2\eta}\,\varepsilon_1^{\frac{1}{2}-\eta} \leq \dfrac{1}{2}$,

ε_2 such that $\left(\dfrac{2}{q}\,\varepsilon_2^{(2-q)/2}\right)^{\frac{1}{q}} \|g\|_{L_p(T_0-\varepsilon_2,T_0)} + \dfrac{2}{1-2\eta}\,\varepsilon_2^{\frac{1}{2}-\eta} \leq \dfrac{1}{4}\ \left(\dfrac{1}{p} + \dfrac{1}{q} = 1\right)$.

Define the interval $J = \{t \in [t_1,T_0) : \varphi(s) \leq L \text{ for } s \in [t_1,t]\}$, where $t_1 = t_1(\varepsilon_1,\varepsilon_2)$ and $L > \varphi(t_1)$. By the continuity of φ the interval $J \neq \emptyset$ is closed in $[t_1,T_0)$. For $t \in J$ we have

$$\varphi(t) \le \varphi(t_1) + \left(\frac{2}{q}(T_0-t_1)^{(2-q)/2}\right)^{\frac{1}{q}} \|g\|_{L_p(T_0,t_1,T_0)} + \frac{2}{1-2\eta}(T_0-t_1)^{\frac{1}{2}-\eta} L^{2(1-\eta)} .$$

If $\varphi(t_1) \le \frac{1}{2}$ we choose $L = 1$. By the choice of ε_2 we have $\varphi(t) < L$ for all $t \in J$ which shows that J is also open in $[t_1,T_0)$. If $\varphi(t_1) > \frac{1}{2}$ we choose $L = 2\varphi(t_1)$. By the choices of ε_1 and ε_2 we have again $\varphi(t) < L$ for all $t \in J$. The closedness and openness of J in $[t_1,T_0)$ implies that $J = [t_1,T_0)$.

We next present an improvement of Lemma 1.2 which we owe to L. Cafarelli.

Lemma A.1. Let $\varphi \in C([t_0,T_0), \mathbb{R}_+)$ satisfy

(i) $\quad 0 \le \varphi(t) \le C + \int_{t_0}^{t} (t-s)^{-\frac{1}{2}-\eta} \varphi(s)^{2(1-\eta)} ds$

for $t \in [t_0,T_0)$ and some $0 \le \eta < \frac{1}{2}$, $C \ge 0$;

(ii) $\varphi \in L_2((t_0,T_0), \mathbb{R}_+)$.

Then $\varphi(t) \le L$ on $[t_0,T_0)$ where L *depends* explicitly on C, $\varphi(t_0)$, $\|\varphi\|_{L_2(t_0,T_0)}$, and η (see (3.4)).

(Observe that t_0 is fixed in this lemma.)

Proof: Define the sets $A_k = \{t \in [t_0,T_0),\ 2^k \le \varphi(t) < 2^{k+1}\}$, $k = 0,1,\ldots$, and for $\lambda \in (0,1)$ $J_\lambda = \{k \in \mathbb{N}_0,\ |A_k| > \lambda 2^{-2k}\}$ ($|A_k|$ denotes the measure of A_k.) Because of (ii) we have

$$|A_k| 2^{2k} \le \int_{A_k} \varphi(s)^2 ds \le \|\varphi\|^2_{L_2(t_0,T_0)}$$

and

$$\mathrm{card}(J_\lambda)\lambda = \sum_{k\in J_\lambda} \lambda < \sum_{k\in J_\lambda} |A_k| 2^{2k} \le \|\varphi\|^2_{L_2(t_0,T_0)} \equiv \|\varphi\|^2 ,$$

which implies

$$|A_k| \leq \|\varphi\|^2 2^{-2k} \quad ,$$

$$\operatorname{card}(J_\lambda) \leq \|\varphi\|^2 \lambda^{-1} \quad (\operatorname{card}(J_\lambda) \text{ denotes the cardinality of } J_\lambda) \ .$$

Now choose

$$\lambda < \frac{1}{4}\left(\frac{1-2\eta}{32}\right)^{\frac{2}{1-2\eta}} \quad ,$$

(3.3)

$$M = \max\left\{\frac{2}{3}(\ln 2)^{-1}\ln \varphi(t_0),\ \frac{1}{2}(1+(\ln 2)^{-1}\ln C),\ (\ln 2)^{-1}\ln\left(\frac{16(2\|\varphi\|)^{1-2\eta}}{1-2\eta}\right)\right\} \ .$$

Define the intervals

$$I_\nu = [2^{M\nu}, 2^{M(\nu+1)}) \quad , \quad \nu = 1,\ldots\ldots,[\|\varphi\|^2\lambda^{-1}]+1 \quad ,$$

where $[\|\varphi\|^2\lambda^{-1}]$ is the biggest integer $\leq \|\varphi\|^2\lambda^{-1}$.

Then there exists a ν_0 such that $I_{\nu_0} \cap J_\lambda = \emptyset$.

We claim that there is no $t \in [t_0,T_0)$ such that $\varphi(t) \geq 2^{M(\nu_0+1)}$.

Assume that there are such t's. Because of $\varphi(t_0) < 2^{2M}$ there is a <u>first</u> $t_1 \in (t_0,T_0)$ such that $\varphi(t_1) = 2^{M(\nu_0+1)}$. By the definition of A_k clearly $[t_0,t_1) \subset \bigcup_{k=0}^{M(\nu_0+1)} A_k$. Now, by (i):

$$2^{M(\nu_0+1)} = \varphi(t_1) \leq C + \int_{t_0}^{t_1}(t_1-s)^{-\frac{1}{2}-\eta}\varphi(s)^{2(1-\eta)}ds \leq C + \sum_{k=0}^{M(\nu_0+1)}\int_{A_k}(t_1-s)^{-\frac{1}{2}-\eta}\varphi(s)^{2(1-\eta)}ds$$

$$\leq C + \frac{2}{1-2\eta}\sum_{k=0}^{M(\nu_0+1)} 2^{2(k+1)(1-\eta)}|A_k|^{\frac{1}{2}-\eta} \ .$$

We split the sum into $\sum_{k=0}^{M\nu_0-1} + \sum_{k=M\nu_0}^{M(\nu_0+1)}$. For the first sum we have the estimate $|A_k|^{\frac{1}{2}-\eta} \leq \|\varphi\|^{1-2\eta}2^{-k(1-2\eta)}$ and for the second, in view of $I_{\nu_0} \cap J_\lambda = \emptyset$,

$|A_k|^{\frac{1}{2}-\eta} \leq \lambda^{\frac{1}{2}-\eta} 2^{-k(1-2\eta)}$. Thus we get:

$$2^{M(\nu_0+1)} \leq C + \frac{4}{1-2\eta}\left\{(2\|\varphi\|)^{1-2\eta}\sum_{K=0}^{M\nu_0-1} 2^k + (2\lambda^{\frac{1}{2}})^{1-2\eta}\sum_{k=M\nu_0}^{M(\nu_0+1)} 2^k\right\}$$

$$\leq C + \frac{4\cdot 2^{1-2\eta}}{1-2\eta}\left\{\|\varphi\|^{1-2\eta}2^{-M} + 2\lambda^{\frac{1}{2}-\eta}\right\} 2^{M(\nu_0+1)}$$

$$< 2^{M(\nu_0+1)}$$

by the choices of λ and M.

This contradiction shows that

$$\varphi(t) < 2^{M(\nu_0+1)} \leq 2^{M(\|\varphi\|^2\lambda^{-1}+1)}, \tag{3.4}$$

where λ and M are given by (3.3).

Remark 9: The assumption that φ is continuous on $[t_0,T_0)$ was only needed to assure that φ is locally bounded on $[t_0,t_0+\delta)$ and that the time t_1 where $\varphi(t_1) = 2^{M(\nu_0+1)}$ for "the first time" is well defined.

Remark 10: The only property of the kernel which was used in both versions of the lemma is the estimate

$$\int_A (t_1-s)^{-\frac{1}{2}-\eta} ds \leq \frac{2}{1-2\eta}|A|^{\frac{1}{2}-\eta} \quad \text{for } A \subset (t_0,t_1) .$$

Thus the kernel could be replaced by any function $k(t,s)$ in the Marcinkiewicz classes

$$M^{\frac{2}{1-2\eta}}(t_0,t) = \{k(t,\cdot):(t_0,t) \to \mathbb{R}\,,\ |\int_A k(t,s)ds| \leq c_{52}|A|^{\frac{1}{2}-\eta}\}$$

for all measurable $A \subset (t_0,t)$, $t \in (t_0,T_0)$ being arbitrary.

Remark 11. Using Lemma A.1 instead of Lemma 1.2 we get an a-priori estimate of $\|u(t)\|_m$ on $[t_0,T_0)$ depending on t_0, T_0, $\|u(t_0)\|_m$, g, and the constant c_{30} in the energy inequality (1.44).

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We prove the global existence (in time) for any solution of an abstract semilinear evolution equation in Hilbert space provided the solution satisfies an energy inequality and the nonlinearity does not exceed a certain growth rate. When applied to semilinear parabolic initial-boundary-value problems the result admits also the limiting growth rates which were given by Sobolevskii and Friedman, but which were not permitted in their theorem. The Navier-Stokes system in two dimensions is a special case of our general result. The method is based on the theories of semigroups and fractional powers of regularly accretive linear operators and on a nonlinear integral inequality which gives the crucial a-priori estimate for global existence.

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